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von

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Contents

Zusammenfassung

Die vorliegende Dissertation befasst sich mit einer mathematischen Analyse von Modellen aus der Nachrichtentechnik, welche thematisch dem Bereich *Wahrscheinlichkeitstheorie und stochastische Prozesse* gemäß der *Mathematics Subject Classification* zugeordnet ist. Der Inhalt dieser Arbeit ist in zwei Kapitel aufgeteilt, die zwei verschiedene und eigenständige Modelle behandeln.

Im ersten Kapitel geht es um Polling-Modelle (*polling models*), welche verwendet werden können, um Kommunikationssysteme in Telekommunikationsnetzen zu modellieren. Das Thema ist der Warteschlangentheorie zugeordnet, einem Gebiet der angewandten Wahrscheinlichkeitstheorie. Angewandte Wahrscheinlichkeitstheorie ist ein umfassendes Forschungsgebiet, das die Anwendung der Wahrscheinlichkeitstheorie auf Fragestellungen aus der Biologie, der Informatik, der Wirtschaftswissenschaft, des Finanzwesens, des Versicherungswesens, der Heilkunde und der Physik abdeckt, um nur ein paar zu nennen.

In Wartesystemen ist der aufkommende Verkehr oft nicht vorhersehbar, seien es die Ankunftszeiten oder die Größen der Anforderungen. Daher können Konflikte um die Nutzung der zur Verfügung stehenden Ressourcen auftreten und sich Warteschlangen bilden. Da die Ungewissheit als Zufall modelliert werden kann, kann man stochastische Modelle entwickeln. Folglich kommen Hilfsmittel der Wahrscheinlichkeitstheorie und Methoden der Warteschlangentheorie zum Tragen, um diese Modelle beispielsweise in Bezug auf dessen Leistungsstärke zu untersuchen.

Das zweite Kapitel beschäftigt sich mit einem Spezialfall der Modelle zufälliger Kugeln (*random balls models* oder *germ-grain models*) und gehört zum Bereich der Zufallsfelder (*random fields*) und der Grenzwertsätze. Ein Zufallsfeld ist ein verallgemeinerter stochastischer Prozess, d.h. eine Familie von Zufallsvariablen. In unserem Fall ist die Indexmenge dieser Familie ein Raum von signierten Maßen. Unsere Absicht ist es, Grenzwertsätze für eine Folge von Zufallsfeldern herzuleiten, welche durch einen Skalierungsparameter indiziert sind. Genauer gesagt sind wir an der schwachen Konvergenz der endlich-dimensionalen Verteilungen interessiert.

Die Motivation für das Untersuchen der Modelle zufälliger Kugeln beruht ebenfalls auf Telekommunikationsnetzen. Zum Beispiel kann Verkehr, der von unabhängigen Quellen über die Zeit erzeugt wird, durch diese Modelle beschrieben werden. Folglich hat solch eine Untersuchung potentielle Anwendungen in der Analyse von Kommunikationsnetzen.

Nach dieser bisherigen thematischen Einordnung, richten wir nun unser Augenmerk konkreter auf die Inhalte. In Kapitel 1 setzen wir uns mit Polling-Modellen im Sinne von Takagi [40] auseinander. Diese Modelle sind zyklische Bediensysteme mit mehreren Warteschlangen in denen ein einzelner Bediener periodisch agiert. Im Gegensatz zu einer klassischen Art und Weise des Bedienens, z.B. *exhaustive*, *gated* oder *limited*, ist die Besonderheit unseres Bedieners, dass er an einer leeren Warteschlange gezwungen werden könnte, untätig auf neue Arbeit zu warten, anstatt zur nächsten Station zu wechseln. Diese erzwungenen Stillstandszeiten erfolgen nach einer vorgegebenen „abwartenden“ Strategie. Wir betrachten vier verschiedene abwartende Strategien, welche diese Wartezeiten regeln: Entweder gibt es ein feststehendes, abzuwartendes Zeitguthaben an jeder Station (Strategie I), oder es gibt eine vorher festgelegte, kleinstmögliche Aufenthaltszeit des Bedieners an jeder Station (Strategie II), oder der Bediener muss mindestens eine feststehende Zeit verweilen nachdem er das erste Mal an der Station untätig wurde (Strategie III), oder der Bediener wartet solange auf neue Arbeit bis ein Zeitgeber abläuft, welcher nur aktiviert wird, wenn der Bediener bei Ankunft eine leere Station auffindet (Strategie IV). Wir halten fest, dass Strategie I ausführlich von Aurzada et al. [4] analysiert wurde und dass Strategie IV für einen Zeitgeber und zwei Stationen von Boxma et al. [10] untersucht wurde.

Wir nehmen an, dass die Ankünfte neuer Arbeit gemäß Poisson-Prozessen erfolgen und wir lassen allgemeine Bedien- und Wechselzeiten zu. Die Resultate sind Formeln für die erwartete mittlere Wartezeit ankommender Arbeit, Charakterisierungen derjenigen Fälle eines Polling-Modells mit zwei Stationen, bei denen die abwartenden Strategien eine kleinere Wartezeit gegenüber der *exhaustive*-Strategie erzielen, und ein Vergleich der Strategien untereinander.

In Kapitel 2 betrachten wir zufällige Rechtecke (auch Boxen genannt) in $\mathbb{R}^2$, die entsprechend eines Poissonschen Zufallsmaßes verteilt sind, d.h. unabhängig und gleichmäßig in der Ebene verstreut sind. Die Verteilungen der Länge und der Breite der Rechtecke sind *heavy-tailed* mit verschiedenen Parametern. Wir untersuchen das Verhalten zugehöriger Zufallsfelder während die Intensität des Zufallsmaßes gegen unendlich geht und die erwarteten Kantenlängen gegen Null konvergieren. Wir behandeln, um genauer zu sein, zentrierte Zufallsfelder, welche das angesammelte Volumen erfassen, das durch die Boxen bedingt wurde. Wir charakterisieren die auftretenden Fälle, die von dem gemeinsamen Verhalten des Skalierungsparameters und der Intensität des Poissonschen Zufallsmaßes abhängen. Des Weiteren bestimmen wir die verschiedenen Grenzprozesse. Die Klasse dieser Grenzprozesse beinhaltet lineare Gaußprozesse, kompensierte Poisson-Integrale und Integrale bezüglich eines stabilen Zufallsmaßes. Wir beleuchten die Grenzprozesse hinsichtlich statistischer Eigenschaften wie Translationsinvarianz, Selbstähnlichkeit und *aggregate-similarity*. Außerdem führen wir eine Abwandlung unseres Modells zufälliger Boxen (*random boxes model*) ein, wobei jede Box zusätzlich zufällig gedreht wird, und stellen fest, dass verschiedene Fälle der Skalierung durch eine graphische Darstellung unterschieden werden können.

Summary

This thesis deals with a mathematical analysis of models from communications engineering, which is thematically located in the field of *probability theory and stochastic processes* according to the Mathematics Subject Classification (MSC). The content of this work is divided into two chapters that address two different and independent models.

The first chapter treats polling models that can be used to model communication systems in telecommunication networks. The topic is part of the area of queueing theory, a field of applied probability. Applied probability is a broad research area that covers the application of probability theory to problems in biology, computer science, economics, finance, insurance, medicine and physics, just to name a few.

In queueing systems, the traffic offered to the system is often not predictable, be it the arrival times or the sizes of the demands. Therefore, conflicts for the use of the available resource may arise and queues may form. Since the uncertainty can be modelled as random, one can develop stochastic models. Then, probabilistic tools and queueing-theoretic methods come into play in order to analyse these models, e.g., with respect to the performance of the systems.

The second chapter is devoted to a special case of random balls models (also known as germ-grain models) and belongs to the area of random fields and limit theorems. A random field is a generalised stochastic process, i.e., a family of random variables. In our case, the index set of this family is a space of signed measures. Our purpose is to obtain limit theorems for a sequence of random fields, which are indexed by a scaling parameter. More precisely, we are interested in the weak convergence of the finite-dimensional distributions.

Motivation for the study of random balls models also comes from telecommunication networks. For example, traffic generated by independent sources over time can be described by these models. Hence, such a study has potential applications in the analysis of communication networks.

After this thematic summary so far, we concentrate our attention on the more detailed content. In Chapter 1, we discuss polling models in the sense of Takagi [40]. These models are multiple queue, cyclic service systems where a single server operates in cyclic order. In contrast to classical service

policies, e.g., exhaustive, gated or limited service, the feature of our server is that it may be forced to wait idly for new jobs at an empty queue instead of switching to the next station. These forced idle times happen according to a given wait-and-see strategy. We look more closely at four different wait-and-see strategies that govern these waiting periods: Either there is a fixed wait-and-see credit at each station (Strategy I), or there is a predetermined minimum sojourn time of the server at each station (Strategy II), or the server has to stay at least a fixed time after becoming idle for the first time at the station (Strategy III), or the server waits for a new job until a timer expires that is only activated if the server finds the station empty upon arrival (Strategy IV). We note that Strategy I is extensively analysed by Aurzada et al. [4] and Strategy IV is examined by Boxma et al. [10] for one timer and two stations.

We assume that arrivals of new jobs occur according to Poisson processes and we allow general service and switchover time distributions. The results are formulas for the mean average queueing delay of a job, characterisations of the cases for a polling model with two stations where the wait-and-see strategies yield a lower delay compared to the exhaustive strategy, and a comparison of the strategies among each other.

In Chapter 2, we consider random rectangles (also called boxes) in $\mathbb{R}^2$ that are distributed according to a Poisson random measure, i.e., independently and uniformly scattered in the plane. The distributions of the length and the width of the rectangles are heavy-tailed with different parameters. We investigate the scaling behaviour of related random fields as the intensity of the random measure grows to infinity while the expected edge lengths tend to zero. To be more precise, we deal with centred random fields that capture the cumulative volume induced by the boxes. We characterise the arising scaling regimes, which depend on the joint behaviour of the scaling parameter and the intensity of the Poisson random measure, and we identify the different limiting random fields. The class of these scaling limits contains linear random fields that are Gaussian, compensated Poisson integrals and integrals with respect to a stable random measure. We examine the limiting random fields with regard to statistical properties like translation invariance, self-similarity and aggregate-similarity. Moreover, we introduce a modification of our random boxes model where each box is additionally randomly rotated, and we observe that different scaling regimes can be distinguished by a graphical representation.

Introduction

Motivation

In this section, we give an application-oriented motivation for our work. Nevertheless, the content of the thesis is a definitely theoretical work because the mathematical aspects of the applied problems are of primary interest to us. We start with motivating the analysis of polling models (using forced idle times).

A polling model is a system of multiple queues that are accessed by a single server in cyclic order. Two of the first papers discussing the performance of a polling model with a single buffer for each queue were published in the 1950s (see [31, 32]). The application was as follows: In the British cotton industry, a single operative walks round the group of machines in a strictly defined order. He repairs any machine which is found stopped and passes any machine which is running.

Later on, with the growing importance of computer communication networks, an extensive study of polling models was carried out, e.g., in order to investigate wide-area networks or token ring schemes.

We look more closely at the motivation coming from communications engineering. We imagine that several nodes in a communication system are connected via some technology, be it wireless, optical fibre, or some other physics. The different nodes communicate via a certain protocol governed by the technology in use. Often, these communication protocols leave certain working parameters open (in our case, the wait-and-see parameters T_i). One question is to ask for the best possible adjustment of the working parameters such that a performance measure like the mean average queueing delay is minimised. Since the traffic generation is not co-ordinated (one can think of the nodes as different sub-networks that communicate with each other), it can be modelled as random.

In this context, a concrete real-world application is the so-called Ethernet Passive Optical Network (EPON) from the area of telecommunication (see, e.g., [26]). This network protocol belongs to the standards IEEE 802.3, which have been produced by a working group of the Institute of Electrical and Electronics Engineers (IEEE). The network carries data traffic in Ethernet frames between a telecommunications company and various customers, more precisely, between the last provider-owned node and the customer's premises. Depending on the point of view, this protocol refers to the 'last mile' and 'first mile', respectively. Let us assume that an optical fibre cable connects the node of the service provider to the end user and the data transmission happens by electromagnetic radiation. Each wavelength channel on the cable can only be operated either upstream (messages are sent from the end user to the service provider) or downstream at a given time. Switching the direction of data transmission causes an idle time (switchover time). Therefore, each channel of an EPON can be regarded as a polling model.

The next application originates from manufacturing and presents a reason for using forced idle times. In cyclic production systems, a machine can process several classes of jobs but it requires a set-up time between the classes, for example, for tool switching. In former times, a hypothesis in production theory was that reduction of set-up times improve performance. However, Sarkar and Zangwill [39] looked at a simple two-queue system and showed numerically that reducing the set-up time in one queue results in an increase of the mean waiting times in both queues. In other words, forced idle times *may* reduce mean waiting times in polling models.

A final example from everyday life concerns road traffic control. Let us imagine that there are road works on a part of a road such that there is only one traffic lane available, i.e., the traffic can only flow in one direction. Since vehicles arrive from both directions, the traffic is controlled by traffic lights and vehicles thus may have to wait. Moreover, the change of signals requires some time. Minimising (expectations of) waiting times by an optimal control of the traffic lights might be one goal. This situation can be represented by a polling model using forced idle times. Different wait-and-see strategies correspond to different classes of traffic lights control. In view of a result in Subsection 1.2.2, it may be recommendable that some vehicles have to wait on one side while the traffic light on the opposite side is still green (even though no vehicle is present there).

We continue with possible applications of a random balls model (the second model treated in this thesis). A long list of references can be found at the beginning of Chapter 3 in [29], which is inspired by the work from Kaj and Taqqu [23]. Let us establish the general connection with telecommunication networks first. It is of interest to know and understand how the traffic looks like at a node of such a network. We assume that sources produce on-off signals. An on-off process consists of alternating periods of activity and inactivity. These periods can be very long since a lot of data may be transmitted in a period of activity or there is no activity for a long time. We note that the use of heavy-tailed distributions in the models originates from this property. The traffic that accumulates at a node comes from *many* of such sources. Therefore, it is natural to ask for limit theorems. Possible limiting processes can be fractional Brownian motions or stable processes, for example.

To be more precise, in a random balls model, one considers the superposition of objects that are uniformly scattered. In dimension one, the model applies to the random variation in network traffic, where the traffic is generated by independent sources over time. One can think of the starting times of transmission periods and the intervals of duration being the locations and the sizes of the balls, respectively. The quantity of interest is the limiting distribution of the aggregated traffic as the time and the number of sources both tend to infinity (possibly with different rates). These different rates can result in different scaling limits of the superposed network traffic.

More recent work investigates extensions of the models, e.g., weighted random balls models or multi-dimensional models. An additional weight of the ball can be seen as the amount of required resources, the transmission power or the file size (cf. [11, 17, 23, 29]). Our random boxes model can be interpreted in the same way, when the length of the box is thought to be transmission time and the width a weight representing, for example, a transmission rate.

Furthermore, random balls models can serve as models for simplified two-dimensional wireless networks. The network consists of stations that are spatially uniformly distributed and equipped with emitters. In our case, the range for transmission (with constant power) of each station is given by a rectangular area and the total power of emission is measured.

From a different point of view, random balls models in two dimensions can also represent black-and-white pictures where the number of balls covering a point corresponds to the grey level of a pixel. In three dimensions, a configuration according to a weighted random balls model can represent a heterogeneous medium where the weight density is interpreted as the mass density.

Outline

The remainder of the thesis is structured as follows:

Chapter 1 deals with the investigation of polling models using forced idle times. In Section 1.1, we introduce the model with all its parameters (Subsection 1.1.1) and define the wait-and-see strategies in detail (Subsection 1.1.2). We outline related work in Subsection 1.1.3. Section 1.2 contains the formulas for the mean average queueing delay (Subsection 1.2.1), the cases where it is worth waiting (Subsection 1.2.2), and a comparison of the strategies (Subsection 1.2.3). All proofs of the results as well as the determination of essential quantities are collected in Section 1.3. Finally, we conclude Chapter 1 with a brief summary of the results and some suggestions for future work (Section 1.4).

In Chapter 2, we study the scaling behaviour of random fields that come from a particular random boxes model. First, we introduce the random boxes model and the related random fields (Subsection 2.1.1), outline related work (Subsection 2.1.2) and give an overview on the different scaling regimes (Subsection 2.1.3). Section 2.2 contains the theorems of convergence to the limiting random fields (subdivided into the different scaling regimes in Subsections 2.2.1–2.2.3, respectively), a limit theorem for the case where the length and the width of the boxes have finite variances (Subsection 2.2.4), and further facts on statistical properties of the limits as well as an extension of the model where the boxes are randomly rotated (Subsection 2.2.5). We collect some preliminaries in Section 2.3 in order to prove the results in Section 2.4. A graphical representation of the low intensity sub-regimes can be found in Section 2.5. At the end, we briefly summarise Chapter 2 and sketch some open problems (Section 2.6).

Chapter 1

Polling models using forced idle times

1.1 Introduction

1.1.1 Model

We investigate a polling model in the sense of Takagi [40] consisting of $N \geq 1$ stations that are served by one server. The stations are labelled by the indices from 1 to N and served in ascending, cyclic order with $N + 1 \triangleq 1$.

Each station i has its own queue, which is fed by jobs generated by a Poisson arrival process with intensity λ_i. From now on, we always refer to these jobs as messages. Each message has a random length (also called service time). The mean and second moment of the message length distribution are assumed to be finite and denoted by b_i and $b_i^{(2)}$, respectively.

An illustration of a polling model with $N = 2$ stations is given in Figure 1.1. The three vertical bars at station 2 represent messages that are waiting in the queue in order to be processed.

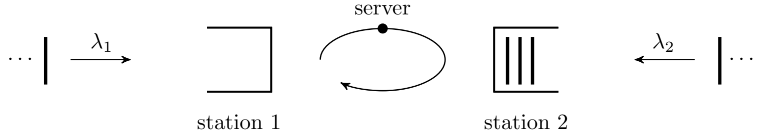

Figure 1.1: Polling model with two stations.

Switching between stations takes a non-negative random idle time, called the switchover time, where the server does not process any messages at any station. The random switchover time R_i from station i to the next station (with distribution function F_{R_i}) is assumed to have finite mean r_i and finite second moment $r_i^{(2)}$. We consider both non-deterministic and deterministic switchover times (in the latter case $r_i^{(2)} = r_i^2$ for $i = 1, \ldots, N$). The sum of the mean switchover times and the second moment of the sum of independent switchover times are denoted by $r_0 := \sum_{i=1}^N r_i$ and $r_0^{(2)} := \sum_{i=1}^N r_i^{(2)} + \sum_{i,j=1,i\neq j}^N r_i r_j$, respectively.

The message generation process, the lengths of the messages, and the switchover times are assumed to be independent (everything among each other and with respect to the other processes and stations).

The goal is to obtain explicit formulas for the mean average queueing delay of a message in a polling model with a given wait-and-see strategy in steady state. The delay is the time a message experiences from the point in time when it arrives in one of the queues until its service starts, i.e., excluding the service time. The expected delay of a message generated at station i is denoted by $\mathbb{E}D_i$. The mean average queueing delay $\bar{D}$ is then defined by

$$\bar{D} := \sum_{i=1}^{N} \frac{\rho_i}{\rho_0} \mathbb{E}D_i,$$

where $\rho_i := \lambda_i b_i$ is the traffic load at station i and $\rho_0 := \sum_{i=1}^N \rho_i$ is the total load offered to the system. We stress that the delays of the different stations are weighted by the traffic intensity ρ_i, which implicitly includes weighting by the mean message lengths, whereas the delays $\mathbb{E}D_i$ do not include weighting the delay of the individual messages with their lengths. The mean average queueing delay, which we often just abbreviate as *delay*, is called the *intensity weighted mean waiting time* by Takagi [40].

1.1.2 Wait-and-see strategies

It is characteristic of many service strategies to prevent that the server spends time idly at a station (while there may be work present at other stations). In contrast, we consider wait-and-see strategies where the server may be forced to wait idly for new messages at an empty queue. An advantage consists in the fact that the server does not switch too often, especially when it is not necessary or not worthwhile. This behaviour can be favourable in order to optimise performance measures, for instance, to minimise the delay,

First, we describe the wait-and-see behaviour of the server in general: The server arrives at a station and starts serving in an exhaustive fashion, i.e., processing all waiting messages and newly arriving messages (first come, first served) until the queue is empty. However, once the station is empty or if the server finds an empty station upon its arrival, the server may not immediately switch to the next station; it rather turns idle for some time in order to wait for possibly newly arriving messages ('wait-and-see'). As soon as a new message arrives, the server starts serving immediately and in an exhaustive fashion. Once finished, the server may again turn idle and wait for new messages.

For each of the wait-and-see strategies that we consider in the following, the behaviour of the server at station i is governed by a fixed, real parameter $T_i \geq 0$, which has different interpretations (see below). Of course, the server is not allowed to be idle if at its present station messages are waiting to be processed. The reason for waiting depends only on the current station in the current cycle, i.e., on the evolution of the traffic at the present station since the server arrived there. The server must not use any information about the current queue status at other stations nor about the future of the arrival process at any station. If $T_i = 0$ holds, the service discipline is exhaustive at station i and there is no state of 'wait-and-see' at station i. If this is the case for all stations, we call it the *exhaustive strategy.*

Now, we specify four different wait-and-see strategies. Strategy I is extensively analysed in [4] and Strategy IV is examined in [10] for $N = 2$ stations and $T_2 = 0$. As far as we know, there are no results in the literature on Strategies II and III.

Strategy I. The server has to wait idly the total time T_i for new messages at station i per cycle. Depending on the arrival process, this credit T_i is spent altogether in one single period or in some periods interleaved by different busy periods.

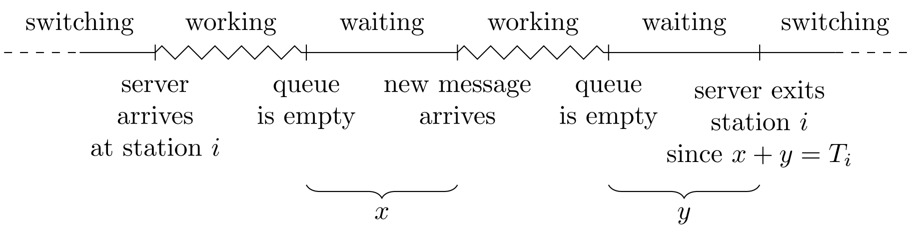

Figure 1.2: Sample for Strategy I.

Figure 1.2 shows an example of a stay of the server at station i in a polling model with Strategy I. In this particular case, there are two waiting periods with lengths x and y, which are interleaved by a busy period. The server exits the station at the point in time as the sum of the waiting periods equals T_i.

Strategy II. The server has to stay at least the minimum sojourn time T_i at station i per cycle. We can regard it as a timer starting upon arrival of the server at this station. Once the server has spent the minimum sojourn time at the station (possibly consisting of several busy and waiting periods), the server exits the station if the queue is empty. If there are still messages waiting or in service as the timer expires, the server continues serving in an exhaustive fashion and switches to the next station as soon as the queue is empty.

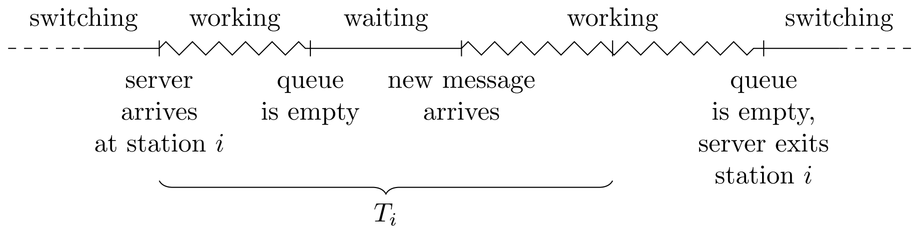

Figure 1.3: Sample for Strategy II.

In the sample in Figure 1.3, there is still work present at station i at the point in time as the timer expires (denoted by the right end of the curly bracket). Therefore, the working period continues until the server becomes idle. We note that the timer started upon arrival of the server at this station because of the definition of Strategy II.

Strategy III. This strategy is a modification of Strategy II. Here, the server is forced to stay at station i at least the fixed time T_i *after* becoming idle for the first time at this station in this cycle. If there are no messages waiting upon arrival of the server, the timer starts immediately as in the case for Strategy II. Otherwise, the timer starts running just after the first busy period.

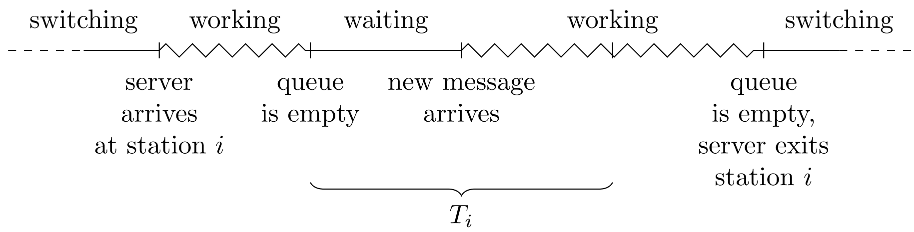

Figure 1.4: Sample for Strategy III.

Figure 1.4 can be regarded as a sample for Strategy III. The timer starts as the queue is empty for the first time at this station (denoted by the left end of the curly bracket). At the point in time as the timer expires (denoted by the right end of the curly bracket), there is still work present and the server continues serving until the queue is empty. In fact, the evolution of the traffic does not differ from the sample in Figure 1.3. However, we emphasise that the timer starts running when the first period of working ends for Strategy III in this sample, i.e., the value of T_i is here less compared to Strategy II.

Strategy IV. The behaviour of the server is based on a case distinction: If there are messages waiting upon arrival of the server, the server starts serving in an exhaustive fashion and then switches to the next station. Otherwise, if the server finds station i empty upon arrival, a timer is activated and the server remains idle for at most the time T_i, waiting for the first arriving message. If the timer expires before the first arrival occurs, the server switches to the next station. Otherwise, if a new message arrives before the timer expires, the server starts serving immediately and in an exhaustive fashion. After this busy period, the server does not wait any longer at this station in the current cycle and switches to the next station.

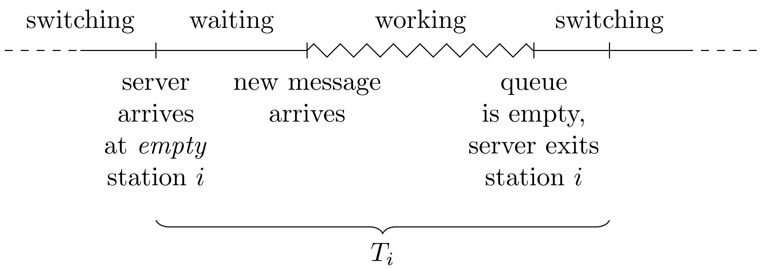

Figure 1.5: Sample for Strategy IV.

In the sample in Figure 1.5, the server finds station i empty upon arrival and the timer thus starts running. The point in time as the timer expires is denoted by the right end of the curly bracket. However, this *exact* point in time is of no importance in the particular sample because there is a working period which is completed before the point in time as the timer expires. Therefore, the server exits the station at the end of the working period due to the definition of Strategy IV.

We stress that we only deal with wait-and-see strategies where timers are deterministic. In order to yield a lower minimal delay, we conjecture that deterministic timers do a better job than random timers. Simulations have also indicated that such an additional randomness (of the timer) in the polling model has no positive effect on the minimal delay.

1.1.3 Related work

First, we refer to Takagi (see, e.g., [40, 41]) for a basic and comprehensive survey on a broad class of polling models.

Aurzada et al. [4] analyse Strategy I and give an explicit formula for the mean average queueing delay in a polling model with N stations. All quantities in the formula can be computed for general distributions of the service times. Furthermore, they characterise several cases where Strategy I yields a lower delay compared to the exhaustive strategy. In these cases, the optimal parameters T_i can be computed explicitly. Finally, they give a lower bound for the delay for a class of wait-and-see strategies that includes Strategies I–IV.

In [10], Boxma et al. discuss a two-queue polling model with a timer as in Strategy IV at station 1, where the timer may be random. They examine different configurations: Either both stations are served exhaustively, or one station is controlled by the 1-limited protocol whereas the other station is served in an exhaustive fashion. The main results are the probability generating function of the queue lengths, expressions for pseudo-conservation laws, and the Laplace transform of the stationary waiting times.

Besides our main references [4] and [10], further papers deal with service strategies which have in common that the server does not necessarily switch to the next station when the current queue is empty.

Li [28] extends the configurations in [10] and considers a polling model with $N = 2$ stations with a timer as in Strategy IV at station 1 but a randomly timed gated mechanism at station 2. Upon arrival at station 2, an exponentially distributed timer is activated. If the server empties the queue before the timer expires, the server exits station 2 immediately. Otherwise, the server exits just after all messages are processed that have an arrival time before the expiration of the timer.

A special case of Strategy IV is given if we set $T_i = \infty$. Whenever the server arrives at an empty station, it waits for the next arriving message at that station. Afanassieva et al. [1] treat this setting and investigate a polling model (which originates from transportation networks) with N stations and several servers that switch according to a routing matrix. Especially a classification of the process concerning recurrence and transience is provided.

Similar to Strategy II, de Haan et al. [14] consider an autonomous server that spends exactly an exponentially distributed amount of time at a station, which implies that service is preemptive. If the service of a customer is interrupted due to this policy, the customer is served again in the next cycle with a new service time generated from the original distribution. The analysis is based on studying embedded Markov chains at specific instants. They give the queue length distributions at various instants and study the approximation of performance measures for multi-queue models using results for single-queue models.

Xie et al. [42] look at polling models with deterministic sojourn times and preemptive service. A representation of the Laplace transform of the workload is given and they obtain bounds as well as an approximation for the mean workload.

Next, we present some works that are devoted to the study of forced idle times where the server is not allowed to resume service immediately when a new message arrives during these idle periods.

Sarkar and Zangwill [39] showed numerically first by an example in a cyclic production system that reduction of the mean switchover time can increase the mean waiting times due to variance effects. From a different perspective, they observe that the increase of switchover times may reduce the mean waiting times in polling models.

Afterwards, Peköz [33] studies a server which remains idle for a fixed deterministic time after becoming idle for the first time at the current station. Immediately after this, the server either switches to the next station or processes the new messages (if present) until the queue is empty again before switching. Results consists of the computation of expected waiting times.

Cooper et al. [12] show that the increase of switchover times by a deterministic forced idle time can reduce the mean waiting time at a station in some cases. They characterise these cases for the exhaustive and gated service discipline, and they give the optimal length of the forced idle time.

Samaddar and Whalen [37] reduce expected waiting times by inserting a variable idle time that is a function of the switchover times. They provide an explicit procedure for finding the optimal variable idle time if the switchover times follow any finite discrete distribution.

In addition to the above-mentioned strategies, there are papers addressing polling models with time-limited service, i.e., messages are processed at a station for a certain period of time or until the queue is empty, whichever occurs first. Often, these time limits are assumed to be exponentially distributed.

Eliazar and Yechiali [16] look at three different configurations. If there is still work at the station when the exponentially distributed timer expires, the server either completes all the present work, or completes only the job which is currently processed, or stops working immediately at this station and switches to the next station. The results include performance measures, pseudo-conservation laws and the expectation of the queue length in steady state.

Leung [27] considers a multiple-queue polling model with non-preemptive service and exponentially distributed time limits for the sojourn time at each station. If a customer is in service as the timer expires, the server continues serving this customer before switching. Using discrete Fourier transforms, queue length and delay distributions are obtained. Moreover, a comparison of waiting times for exponential time limits, constant time limits and k-limited service policies can be found there.

In [2], Al Hanbali et al. deal with time-limited preemptive service systems with Poisson batch arrivals and phase-type service times. If the exponentially distributed timer expires during the service of a customer, this service is stopped and a new service time is redrawn for the next cycle. They use an iterative scheme in order to get the joint queue length distribution at server departure instants.

We refer to de Haan [13] for further random time limits and continue with deterministic time limits. The latter ones are studied by de Souza e Silva et al. [15]. They use embedded Markov chains, develop an algorithm for the transition probabilities and are able to compute performance measures.

Frigui and Alfa [19] also investigate such a time-limited polling model with a Markovian arrival process. Here, the service times have discrete phase-type distributions. They make use of an iterative procedure to compute the queue length distribution and the average waiting time.

The purpose of a more general work from Liu et al. [30] is to find optimal polling policies that stochastically minimise the unfinished work in the system. This problem is decomposed into sub-problems such that they can obtain optimal policies for some particular polling models.

1.2 Results

In this section, we give formulas for the mean average queueing delay that allow us to compute the delay for the different wait-and-see strategies. We further characterise the cases where it is favourable (in the sense of a lower delay) to possibly wait at a station instead of switching. Finally, we compare the different strategies among each other. From now on, we assume that the stability condition $\rho_0 < 1$ of the polling model holds.

1.2.1 Main results

Theorems 1 and 2 provide formulas for the mean average queueing delay in terms

- of the system parameters λ_i, b_i, $b_i^{(2)}$, r_i, $r_i^{(2)}$ for $i = 1, \dots, N$ and
- of the parameter-dependent quantities $(f_i, w_i, \widetilde{r}_i)_{i=1,\dots,N}$ of expectations in steady state which are defined in the following paragraph and which vary depending on the wait-and-see strategy including the parameters T_i. Specifying these expectations for Strategies II–IV in Subsection 1.3.2 is one key novelty..

We define by f_i the expected time per cycle which the server waits at station i. We use $f_0 \coloneqq \sum_{i=1}^{N} f_i$ for the total expected waiting time of the server per cycle (i.e., idle times without switchover times). The expected backward recurrence time (expected spent time) w_i is defined by

$$w_i \coloneqq \mathbb{E}[\text{time since server arrived at station } i \mid \text{server is idle at station } i],$$

i.e., the expectation of the elapsed time since arriving at station i at a random point in time while waiting at station i. Furthermore, we recall the random switchover time R_i from station i to the next station and define the conditional mean switchover time $\widetilde{r}_i$ by

$$\widetilde{r}_i \coloneqq \mathbb{E}[R_i \mid \text{server is idle at station } i+1].$$

This means: Given a random point in time while waiting at station $i+1$, the quantity $\widetilde{r}_i$ is the expected length of the preceding switchover time.

Theorem 1. *The mean average queueing delay of a message in a polling model with Strategy III is given by*

$$\begin{aligned}\bar{D} = {} & \frac{\sum_{i=1}^N \lambda_i b_i^{(2)}}{2(1-\rho_0)} + \frac{(r_0+f_0)\left(\rho_0^2 - \sum_{i=1}^N \rho_i^2\right)}{2\rho_0(1-\rho_0)} + \frac{\frac{\rho_0 r_0^{(2)}}{2} + r_0 \sum_{i=1}^N f_i(\rho_0-\rho_i)}{\rho_0(r_0+f_0)} \\ & + \frac{1}{\rho_0(r_0+f_0)} \left(\sum_{i=1}^N f_i w_i(\rho_0-\rho_i) + \sum_{1\le i<j\le N} f_i f_j (\rho_0-\rho_i-\rho_j) \right) \\ & - \frac{\sum_{i=1}^N f_i \rho_i (\rho_0-\rho_i)}{\rho_0(1-\rho_0)}.\end{aligned} \tag{1.1}$$

The quantities f_i and w_i for $i = 1, \dots, N$ are specified for exponentially distributed service times in Subsection 1.3.2.

We refer to [4] for the delay of a message in a polling model with Strategy I. We note that the formula in Theorem 1 also applies to Strategy I but with different values for the parameter-dependent quantities f_i and w_i for $i = 1, \dots, N$. For Strategies II and IV, we restrict the number of stations to $N = 2$ due to the technical effort that would be required otherwise to compute further parameter-dependent quantities which would arise in the formula for the delay.

Theorem 2. *The mean average queueing delay of a message in a polling model with $N = 2$ stations and Strategy II as well as Strategy IV is given by*

$$\begin{aligned}\bar{D} = {} & \frac{\sum_{i=1}^2 \lambda_i b_i^{(2)}}{2(1-\rho_0)} + \frac{r_0\rho_1\rho_2}{\rho_0(1-\rho_0)} + \frac{r_0^{(2)}}{2(r_0+f_0)} \\ & + \frac{\rho_2 f_1}{\rho_0(r_0+f_0)}(r_1 + \widetilde{r}_2 + w_1) \\ & + \frac{\rho_1 f_2}{\rho_0(r_0+f_0)}(\widetilde{r}_1 + r_2 + w_2).\end{aligned} \tag{1.2}$$

The quantities f_i, w_i and $\widetilde{r}_i$ for $i = 1, 2$ are specified in Subsection 1.3.2 (in the case of Strategy II only for exponentially distributed service times).

We stress that formulas (1.1) and (1.2) are valid for general distributions of the service times. However, we emphasise that for Strategies II and III we are only able to compute the quantities $(f_i, w_i, \widetilde{r}_i)_i$ explicitly for exponentially distributed service times because formula (1.7) below only exists for the M/M/1 queue in the literature.

1.2.2 Is it worth waiting?

In addition to the formulas for the delay, Theorems 1 and 2 allow us to put the following question: Given the system parameters, how does one have to adjust the parameters $T_i \geq 0$ such that the delay is minimised. In this context, we regard the delay $\bar{D}$ for $N = 2$ as a function of (T_1, T_2) and thus write $\bar{D}(T_1, T_2)$. In general, we cannot compute a minimiser of this problem

$$\min_{T_1 \geq 0,\, T_2 \geq 0} \bar{D}(T_1, T_2)$$

for Strategies II–IV analytically. Nevertheless, we obtain necessary and sufficient conditions for these wait-and-see strategies in a polling model with exponentially distributed service times such that it is favourable to wait in comparison to the exhaustive strategy, i.e., there exists some $(T_1, T_2) \neq (0, 0)$ such that $\bar{D}(T_1, T_2) < \bar{D}(0, 0)$.

We note that we only consider the two cases with the additional restriction $T_2 = 0$ and $T_1 = T_2$, respectively. To anticipate the result, one can say that the benefit of waiting arises from the asymmetry of the system or from non-deterministic switchover times.

Theorem 3. *Let $N = 2$ and $T_2 = 0$. It is worth waiting at station 1, i.e., there exists some $T_1 > 0$ such that $\bar{D}(T_1, 0) < \bar{D}(0, 0)$, in a polling model with*

- *Strategy III if and only if*

$$\frac{r_0^{(2)}}{2r_0^2} - \frac{\rho_2(1 - \rho_2)}{\rho_0(1 - \rho_0)} > 0, \tag{1.3}$$

- *Strategy II as well as Strategy IV if and only if*

$$\frac{r_0^{(2)}}{2r_0\left(r_1 + \widetilde{r}_2^{\mathrm{IV}}\right)} - \frac{\rho_2}{\rho_0} > 0, \tag{1.4}$$

  where the quantity $\widetilde{r}_2^{\mathrm{IV}}$ is defined just above Theorem 1 and can be computed as in (1.21) below. In the case of deterministic switchover times R_i for $i = 1, 2$, inequality (1.4) simplifies to $\rho_1 > \rho_2$.

We mentioned above that the parameter-dependent quantities f_i, w_i and $\widetilde{r}_i$ can vary depending on the wait-and-see strategy including the parameters T_i. The dependence on the strategy is indicated by a superscript (if it is necessary in the context), for instance, as with $\widetilde{r}_2^{\mathrm{IV}}$ in Theorem 3. However, we note that condition (1.4), which contains $\widetilde{r}_2^{\mathrm{IV}}$, *must not* depend on T_1. Indeed, the quantity $\widetilde{r}_2^{\mathrm{IV}}$ does not depend on the parameters T_i for $i = 1, 2$ (see (1.21) below).

Remark 4. First, we introduce some temporary notation. The coefficient of variation c_X is defined as the ratio of the standard deviation and the mean of a random variable X. We further denote by R_0 the sum of the switchover times. We can observe from Theorem 3 that for c_{R_0} sufficiently large, it is even worth waiting at station 1 in spite of a lower traffic load $\rho_1 < \rho_2$. As a consequence of this, we can make a conjecture for a polling model with $N = 2$ stations and without any restriction on T_i for $i = 1, 2$: In the case of c_{R_0} being sufficiently large, it is favourable to have *positive* parameters T_i at *both* stations instead of just allowing 'wait-and-see' at the station with greater traffic load.

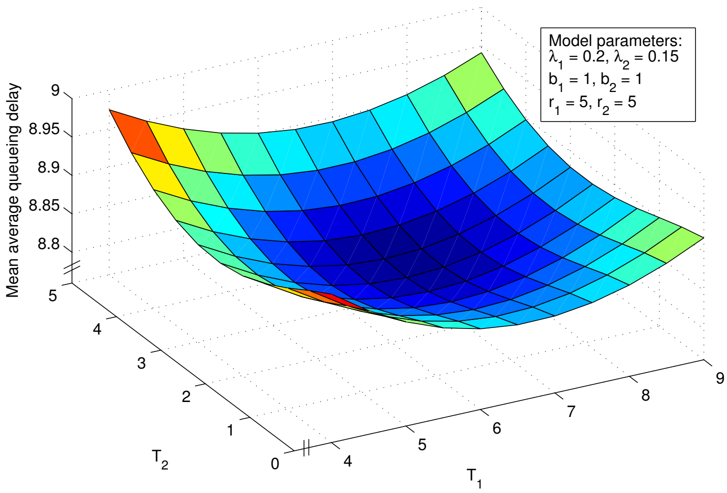

Figure 1.6: Delay for Strategy II as a function of both parameters T_i.

Referring to Remark 4, Figure 1.6 illustrates the delay in a polling model with Strategy II with exponentially distributed switchover times where the traffic load at station 1 is greater than at station 2. For the model parameters chosen there, we observe that waiting at both stations with different parameters T_i yields the lowest delay. The data was computed with the formula from Theorem 2 and the values for $(f_i, w_i, \widetilde{r}_i)_i$ from Subsection 1.3.2.

Similar to above, we get necessary and sufficient conditions for a symmetric polling model with $\rho_1 = \rho_2$ and the restriction $T_1 = T_2$ such that the delay is lower than for the exhaustive strategy. The arrival rates, message length and switchover time distributions are also assumed to be identical for both stations for Strategies II and IV but we can omit this requirement for Strategy III.

Theorem 5. *Let $N = 2$. It is worth waiting with the restriction $T_1 = T_2$, i.e., there exists some $T_1 > 0$ such that $\bar{D}(T_1, T_1) < \bar{D}(0,0)$, in a symmetric polling model with*

- *Strategy III if and only if*
$$\frac{r_0^{(2)}}{r_0^2} - \frac{1-\rho_1}{1-\rho_0} > 0$$
(that can only be satisfied for non-deterministic switchover times),
- *Strategy II as well as Strategy IV if and only if the switchover times are non-deterministic.*

We note that the conditions in Theorems 3 and 5 for Strategy III are identical with those for Strategy I (cf. Theorems 3 and 4 in [4]). According to Remark 5 in [4], we can also extend the assertion for Strategy I in an asymmetric polling model to non-deterministic switchover times.

1.2.3 Comparison of the wait-and-see strategies

We start with an immediate consequence of Theorems 3 and 5. In order to prove the following corollary, one just has to set the system parameters such that the condition (inequality) in Theorems 3 or 5 is fulfilled for Strategies II and IV but not for Strategy III.

Corollary 6. *There are parameter settings of a polling model with $N = 2$ stations where Strategies II and IV yield a lower delay than Strategies I and III, i.e., it is only worth waiting with Strategies II and IV.*

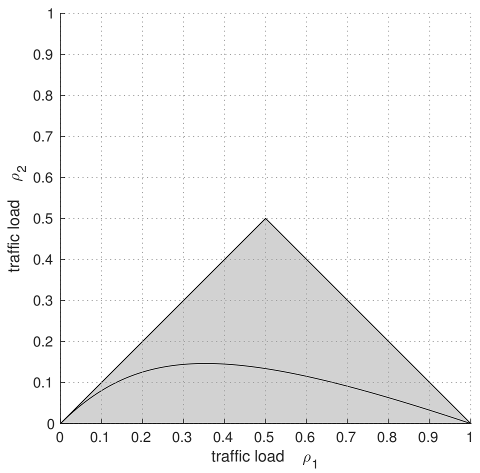

Figure 1.7: Illustration of the conditions in Theorem 3 in case of deterministic switchover times (see explanation in the following paragraph).

Figure 1.7 shows the feasible traffic loads (ρ_1, ρ_2) for Theorem 3 such that it is worth waiting at station 1 in a polling model with deterministic switchover times. Due to the deterministic switchover times, condition (1.3) simplifies to

$$\rho_1 - \rho_1^2 + \rho_2^2 - \rho_2 - 2\rho_1\rho_2 > 0.$$

The filled (grey) area represents the feasible set of traffic loads (ρ_1, ρ_2) such that it is worth waiting with Strategy II as well as Strategy IV, whereas only the area below the curved line belongs to the feasible set in the case of a polling model with Strategy III.

In Figure 1.8, we provide a typical relation between the delays for all four wait-and-see strategies. We consider a polling model with $N = 2$ stations where the server is not allowed to wait idly at station 2. The switchover times are deterministic, symmetrically split among the switchovers and the service times are exponentially distributed. The delay for Strategy I was obtained by the formula in Corollary 2 in [4] and we used formulas (1.1), (1.2) as well as the values for $(f_i, w_i, \widetilde{r}_i)_i$ from Subsection 1.3.2 to compute the data for Strategies II–IV.

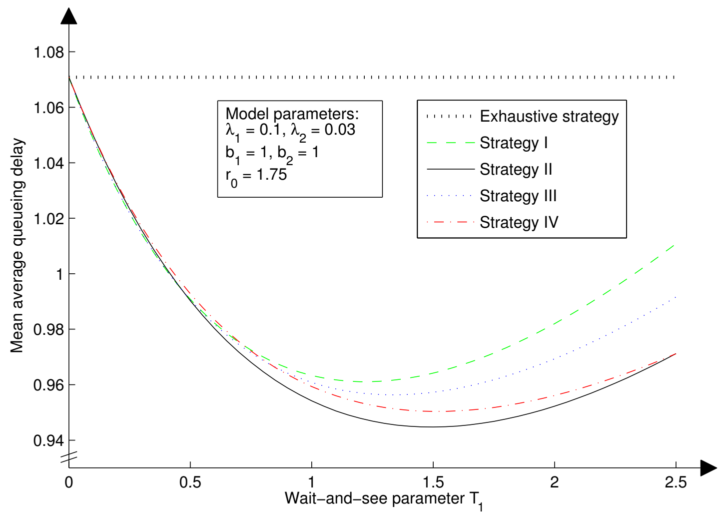

Figure 1.8: Comparison of the delays for the strategies in the case of $T_2 = 0$.

We make the following three observations relating to Figure 1.8: First, all four wait-and-see strategies yield a lower minimal delay compared to the exhaustive strategy. Consequently, the model parameters satisfy the inequalities given in Theorem 3. We can also check this easily using the illustration in Figure 1.7.

Second, we notice that the minimum delays for Strategies I–IV are attained at different wait-and-see parameters T_1^*, respectively. This is due to the particular interpretations of T_1 (cf. definitions of Strategies I–IV in Subsection 1.1.2). We comment on the order of T_1^* for the different strategies from left to right. The parameter for Strategy I captures only waiting periods. The busy periods between waiting periods are also taken into account for Strategy III and the parameter for Strategy II covers on top of that the first busy period at the station as well. We recall that the timer for Strategy IV is only activated if the server finds the station empty upon arrival. Moreover, there is at most one waiting period during a stay of the server at the station. Therefore, the value of T_1 for Strategy IV has to be even greater than for Strategy II (in order that the server spends on average the same time at the station for both strategies).

Third, the ranking of the wait-and-see strategies with respect to the minimal delay observed in Figure 1.8 can be explained naturally: In the best case, the server exits the current station as soon as there is enough work waiting at the other station. Since the server does not have any information about the queue status at the other station, the sojourn time at the current station is the crucial quantity in order to estimate the workload generated at the other station. Hence, there is an optimal sojourn time for each station, and the minimal delay is attained if the expected sojourn time agrees with the optimal sojourn time as good as possible, i.e., with small variance.

Based on this conjecture, we continue with one-to-one comparisons in the following paragraphs:

Strategy I vs. Strategy III. We can observe that Strategies I and III behave similarly. We recall that the conditions in Theorems 3 and 5 for Strategy III are identical with those for Strategy I (cf. Theorems 3 and 4 in [4]). Nevertheless, the minimal delays of both strategies differ in general (cf. Figure 1.8). Relating to the conjecture above, Strategy III seems to allow a lower minimal delay because it takes advantage of the 'information' of the lengths of busy periods between waiting periods while Strategy I does not. The random lengths of these busy periods entail a greater variance of the point in time when the server really switches to the next station in the case of Strategy I.

Strategies I and III vs. Strategy II. Strategy II seems to be better than Strategies I and III. Additionally, the 'information' of the length of the first busy period is implicitly used for 'deciding' when to exit the station. Hence, the variance of the sojourn time at a station for Strategy II is less compared to Strategies I and III. A further advantage can be seen in the following case: We consider a polling model with Strategy II and imagine that there is a sufficiently large first busy period at station 1 such that no waiting period follows in spite of a positive parameter T_1. One can agree that such an additional waiting period would be more trouble than it would be worth from the point of view of waiting messages at station 2. Moreover, a sufficiently large first busy period at station 1 guarantees (on average) that enough work is produced at station 2 in order to make immediate switching useful. In contrast to Strategy II, a positive parameter T_1 in polling models with Strategies I and III would, however, cause such an unfavourable waiting period.

Strategies I and III vs. Strategy IV. According to Corollary 6, there are parameter settings of a polling model where Strategy IV is better than Strategies I and III. This ranking can also be seen in the example in Figure 1.8. Nevertheless, we have to look more closely at Strategy IV. In contrast to the observation so far, a case where Strategy IV is worse than Strategy I is provided in [4]. This is due to the fact that, differently from Strategies I and III, a waiting period can only occur upon arrival at an empty station and there is at most one waiting period for Strategy IV. Especially for a large asymmetry between the stations, more than only one waiting period per cycle may be recommendable, which cannot be put into practice by Strategy IV.

Strategy II vs. Strategy IV. At first glance, Strategies II and IV behave similarly due to Theorems 3 and 5. However, besides the argument concerning the variance of the sojourn time, the disadvantage of Strategy IV always becomes visible when more than only one waiting period per cycle is recommendable (as already mentioned in the preceding paragraph). Therefore, we conjecture that Strategy II is better than Strategy IV.

All in all, we conjecture that Strategy III always yields a lower minimal delay compared to Strategy I and that Strategy II is the best of the investigated wait-and-see strategies. *How much* can be gained by wait-and-see strategies varies depending on the system parameters. It seems to be even possible to give examples where the ratio between the delay in a polling model with forced idle times and the delay for the exhaustive strategy is arbitrarily small.

1.3 Proofs

1.3.1 Proofs of the main results

We show how to derive Theorems 1 and 2, which are based on a decomposition principle from [9] and on the technique of the proofs of Theorems 1 and 8 in [4]. We mention that the proofs of Theorems 1 and 2 are rather standard and that the key novelty is the determination of the parameter-dependent quantities in Subsection 1.3.2.

Proof of Theorem 1: First, we recall some important identities: The cycle time is the time that the server takes from its arrival at station 1 to the next arrival at this station. The mean cycle time in steady state is denoted by $\mathbb{E}C$ and given by

$$\mathbb{E}C = \frac{r_0 + f_0}{1 - \rho_0}.$$

Indeed, this can be seen by looking at the expected time which the server is idle per cycle. This expectation equals the sum of all the mean switchover and waiting times, i.e., $r_0 + f_0$. From a different perspective, we can represent this expected idle time by using the total load offered to the system, which results in $(1 - \rho_0)\mathbb{E}C$.

Next, we refer to a workload decomposition principle in [9, 10] that also holds for our system. We omit a proof since it is analogous to the proofs there. As a consequence of this decomposition principle, we obtain

$$\mathbb{E}V = \mathbb{E}V^{\mathrm{M/G/1}} + q\mathbb{E}V^{\mathrm{switching}} + (1-q)\mathbb{E}V^{\mathrm{waiting}}, \tag{1.5}$$

where $q := \mathbb{P}(\text{server is switching} \mid \text{server is idle})$ and V is the workload at a random point in time in steady state. The workload consists of the sum of all message lengths that are present in the system including the remaining service time of the currently processed message. The quantities $V^{\mathrm{M/G/1}}$ and $V^{\mathrm{switching}}$ (V^{waiting}) refer to the workload in the same polling model without switchover and waiting times, and to the workload given that the server is switching (waiting) at a random point in time, respectively. Furthermore, we can determine the expected workload differently by

$$\mathbb{E}V = \sum_{i=1}^{N} b_i \mathbb{E}[\text{number of messages in queue at station } i] + \sum_{i=1}^{N} \rho_i \frac{b_i^{(2)}}{2b_i},$$

where the first sum accounts for the messages that are not yet in service and the second sum for the currently processed message. The quotient is the expected residual service time of a currently processed message at station i (see (5.15) in [25]). Using Little's law, this equation can be rearranged into

$$\mathbb{E}V = \rho_0 \bar{D} + \sum_{i=1}^{N} \rho_i \frac{b_i^{(2)}}{2b_i}. \tag{1.6}$$

Therefore, we can combine (1.5) and (1.6) in order to obtain a representation of the delay $\bar{D}$. The quantity $\mathbb{E}V^{\mathrm{M/G/1}}$ is given in the literature (see [3, p. 206]).

From now on, we investigate the expected workload present while switching ($\mathbb{E}V^{\mathrm{switching}}$) and waiting ($\mathbb{E}V^{\mathrm{waiting}}$). The former one does not directly depend on the given wait-and-see strategy so that we can proceed in the same way as in [4]. In contrast, the particular wait-and-see strategy influences the expected workload present while waiting. It remains to derive the general formula for $\mathbb{E}V_i^{\text{waiting}}$, which is the expected workload that is present in the system at a random point in time while the server is waiting at station i. Following the computation in [4] (see equation (23) there), we obtain

$$\begin{aligned}
\mathbb{E}V_i^{\text{waiting}} = & \sum_{j<i} r_j \left(\sum_{l=i+1}^{N} \rho_l + \sum_{l=1}^{j} \rho_l \right) + \sum_{j>i} r_j \sum_{l=i+1}^{j} \rho_l \\
& + \sum_{j<i} \rho_j \mathbb{E}C \left(\sum_{l=i+1}^{N} \rho_l + \sum_{l=1}^{j-1} \rho_l \right) + \sum_{j>i} \rho_j \mathbb{E}C \sum_{l=i+1}^{j-1} \rho_l \\
& + \sum_{j<i} f_j \left(\sum_{l=i+1}^{N} \rho_l + \sum_{l=1}^{j-1} \rho_l \right) + \sum_{j>i} f_j \sum_{l=i+1}^{j-1} \rho_l \\
& + (\rho_0 - \rho_i) w_i,
\end{aligned}$$

where w_i denotes the expectation of the elapsed time since arriving at station i at a random point in time while waiting at station i. Combining all the relevant equations, we get formula (1.1) for the delay. $\square$

Proof of Theorem 2: For $N = 2$, the only part that differs from the proof of Theorem 1 is the computation of $\mathbb{E}V_i^{\text{waiting}}$ for $i = 1, 2$. We continue with the discussion of this quantity for $i = 1$ and thus have to keep the condition in mind that the server is currently waiting at station 1. Let us assume that the server is at such a point in time. The present workload has not been generated at station 1 (otherwise the server would not be currently waiting at this station). Therefore, the workload which is currently present can only consist of messages that have been generated at station 2 since exiting that station. The expectation of the elapsed time since exiting station 2 is the sum of the conditional mean switchover time $\widetilde{r}_2$ and the expected backward recurrence time w_1, whose definitions can be found just above Theorem 1. We get

$$\mathbb{E}V_1^{\text{waiting}} = \rho_2(\widetilde{r}_2 + w_1)$$

and for $\mathbb{E}V_2^{\text{waiting}}$, we just have to reverse the roles of 1 and 2. $\square$

Remark 7. The conditional mean switchover time $\widetilde{r}_i$ only differs from r_i for a non-deterministic switchover time for Strategies II and IV. In the case of deterministic switchover times for Strategies II and IV, and in the case of a polling model with Strategies I and III (a waiting period at station i exists every cycle if $T_i > 0$ and there is no dependence on the switchover times), we have $\widetilde{r}_i = r_i$.

Remark 8. We briefly refer to Theorem 8 in [4], which provides a lower bound for the delay for a class of wait-and-see strategies (including Strategies I–IV). The bound given there is correct for a polling model with $N = 2$

stations and deterministic switchover times. In the case of $N = 2$ and non-deterministic switchover times, we can replace r_k (in (35) there) by the mean of the switchover time R_k from station k to the next station given that there is no message arriving at station $k + 1$ while switching. For instance, this quantity equals $\mathbb{E}[R_2 \mid B_0]$ given in (1.21) below for $k = 2$. The replacement is necessary due to the fact that non-deterministic switchover times *can* have an influence on the existence of a waiting period at the next station (cf. Remark 7). In addition for a polling model with $N > 2$ stations, one has to give lower bounds for further quantities which arise in the proof there.

1.3.2 Determination of parameter-dependent quantities

The general formulas for the delay in Theorems 1 and 2 require the specification of $(f_i, w_i, \widetilde{r}_i)_i$ according to the wait-and-see strategy. We recall that the definitions of these quantities can be found just above Theorem 1 and mention that we restrict the service times of the messages to exponential distributions with parameter $\mu_i := 1/b_i$ at station i for $i = 1, \ldots, N$ for Strategies II and III. After the following preparations, we discuss the different wait-and-see strategies separately, where some terms from [10] can also be spotted here for Strategy IV. For the sake of simplicity, we treat Strategy III first.

Preparations

It is helpful to introduce c_i the expected time per cycle which the server spends at station i. This expression is directly related to the mean cycle time $\mathbb{E}C$ and to the expected waiting time f_i at station i by the equation

$$c_i = \rho_i \mathbb{E}C + f_i.$$

We also define $c_0 := \sum_{i=1}^{N} c_i$ and obtain $\mathbb{E}C = c_0 + r_0$.

Furthermore, we require a time-dependent state probability (denoted by $P_{j,k}(x)$) to analyse the delay for Strategies II and III, and we require the distribution of the length of a busy period to analyse the delay for Strategies II and IV.

The probability $P_{j,k}(x)$. According to [25, pp. 53–78], we denote by $P_{j,k}(x)$ the probability that the queue length (including the possibly currently processed message) of an M/M/1 queue with arrival rate λ and service rate μ is k at time x given that the queue length is j at time zero. We introduce the abbreviation $a := 2\mu\sqrt{\rho}$, where the traffic load ρ equals λ/μ, and the modified Bessel functions I_k of the first kind of order k, which can be defined by

$$I_k(x) := \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{k+2m}}{(k+m)!m!}, \quad \text{for } k \in \mathbb{N}_0,$$

and $I_{-k}(x) \coloneqq I_k(x)$ for $k \in \mathbb{N}$. Finally, we have

$$P_{j,k}(x) = e^{-(\lambda+\mu)x}\Bigg(\rho^{(k-j)/2} I_{k-j}(ax) + \rho^{(k-j-1)/2} I_{k+j+1}(ax) \\ + (1-\rho)\rho^k \sum_{l=k+j+2}^{\infty} \rho^{-l/2} I_l(ax)\Bigg) \tag{1.7}$$

due to [25, p. 77]. We emphasise that we have to set $\lambda \coloneqq \lambda_i$ and $\mu \coloneqq \mu_i$ if we look at station i. Hence, the probability $P_{j,k}(x)$ can differ depending on i but we omit such an additional index because it arises out of the context.

The density g_i of a busy period. The density of the length of a busy period at station i is denoted by g_i and the n-fold convolution of g_i with itself by $g_i^{(*n)}$. We get

$$g_i(x) = \sum_{n=1}^{\infty} e^{-\lambda_i x} \frac{(\lambda_i x)^{n-1}}{n!} b_i^{(*n)}(x), \quad \text{for } x \geq 0,$$

from [25, p. 226]. By abuse of notation, $g_i^{(*0)}$ represents the Dirac delta function according to the fact that the length of 0 busy periods is zero. The density $b_i^{(*n)}$ is the n-fold convolution of the service time with itself. For exponentially distributed service times, we obtain the density

$$b_i^{(*n)}(x) = \frac{\mu_i^n x^{n-1}}{(n-1)!} e^{-\mu_i x}, \quad \text{for } x \geq 0,$$

of the Erlang(n, μ_i) distribution, which can also be identified as a gamma distribution. In this particular case, an alternative representation of g_i using the modified Bessel function I_k of the first kind of order $k = 1$ is given in [25, p. 215].

Strategy III

We denote by $q_i(T_i)$ the expected number of messages (including the possibly currently processed message) present at station i after time T_i given that there is no message present at time zero. We use the probability $P_{0,k}(T_i)$ from (1.7) to get

$$q_i(T_i) = \sum_{k=0}^{\infty} k P_{0,k}(T_i).$$

Since we only require the expected number of messages at time T_i, we define the short version $q_i \coloneqq q_i(T_i)$.

The expected sojourn time c_i. For each station, we get the equation

$$c_i = \lambda_i(r_0 + c_0 - c_i)\frac{b_i}{1-\rho_i} + T_i + q_i\frac{b_i}{1-\rho_i}, \tag{1.8}$$

which can be seen as follows: First, the time which the server spends at station i depends on the elapsed time since exiting this station in the preceding cycle up to the current arrival at this station. This expected intervisit time of the server at station i is

$$\mathbb{E}C - c_i = r_0 + c_0 - c_i$$

and the quotient $b_i/(1-\rho_i)$ is the expected length of a busy period (which is caused by *one* arriving message). In order to obtain this latter quantity, we refer to the short calculation using Laplace transforms in [25, pp. 211–213]. Together with the arrival rate λ_i, we can compute the expected length of the first busy period (generated by the waiting messages) at station i and get

$$\lambda_i(r_0 + c_0 - c_i)\frac{b_i}{1-\rho_i}. \tag{1.9}$$

After the first busy period, the server has to spend the time T_i at this station (which can consist of several busy and waiting periods). Then, the server exits the station if the queue is empty at time T_i. Alternatively, if there are messages present at time T_i, the server continues processing messages until the queue is empty. This additional time depends on the expected number q_i of present messages and equals $q_i b_i/(1-\rho_i)$ in expectation.

Using (1.8), we can set up a linear system of equations $Mc = d$ with column vector $c \coloneqq (c_i)_i$, where

$$M \coloneqq \begin{pmatrix} 1 & -\frac{\rho_1}{1-\rho_1} & \cdots & -\frac{\rho_1}{1-\rho_1} \\ -\frac{\rho_2}{1-\rho_2} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\rho_{N-1}}{1-\rho_{N-1}} \\ -\frac{\rho_N}{1-\rho_N} & \cdots & -\frac{\rho_N}{1-\rho_N} & 1 \end{pmatrix}$$

and

$$d \coloneqq \begin{pmatrix} r_0\frac{\rho_1}{1-\rho_1} + T_1 + q_1\frac{b_1}{1-\rho_1} \\ \vdots \\ r_0\frac{\rho_N}{1-\rho_N} + T_N + q_N\frac{b_N}{1-\rho_N} \end{pmatrix}.$$

For instance in the case of two stations, we obtain

$$\begin{aligned} c_1 &= \frac{r_0\rho_1 + (1-\rho_2)((1-\rho_1)T_1 + \rho_1 T_2 + q_1 b_1) + \rho_1 q_2 b_2}{1-\rho_0}, \\ c_2 &= \frac{r_0\rho_2 + (1-\rho_1)((1-\rho_2)T_2 + \rho_2 T_1 + q_2 b_2) + \rho_2 q_1 b_1}{1-\rho_0}. \end{aligned} \tag{1.10}$$

The expected backward recurrence time w_i. The expectation w_i is the sum of two terms: The first summand is the expected length of the first busy period at station i (see term (1.9)). The second summand is the expectation of the elapsed time since becoming idle at station i for the first time at a random point in time while waiting at this station. Therefore, we get

$$w_i = \lambda_i(r_0 + c_0 - c_i)\frac{b_i}{1-\rho_i} + \frac{\int_0^{T_i} xP_{0,0}(x)\mathrm{d}x}{\int_0^{T_i} P_{0,0}(x)\mathrm{d}x}, \tag{1.11}$$

where a random point in time while waiting has the density

$$\frac{P_{0,0}(x)}{\int_0^{T_i} P_{0,0}(y)\mathrm{d}y}, \quad \text{for } x \in [0, T_i].$$

The conditional mean switchover time $\widetilde{r}_i$. Actually, the conditional mean switchover time $\widetilde{r}_i$ does not appear in Theorem 1. Nevertheless, for the sake of completeness, we mention that $\widetilde{r}_i = r_i$ (cf. Remark 7). This is due to the fact that there is a waiting period at station i every cycle for $T_i > 0$ because of the definition of Strategy III. The switchover times do neither have an influence on the existence of a waiting period at the next station nor on the particular evolution of the traffic at the next station *after* the first busy period at that station.

Strategy II

We investigate the steady-state probabilities $\pi_n^{(i)}$ for all $n \in \mathbb{N}_0$ that the server finds n messages waiting upon arrival at station i. We consider deterministic switchover times in this paragraph first. The following system of equations describes the relation of consecutive visits at the stations. The probability of finding n messages upon arrival at station 1 depends on the intervisit time of the server, i.e., the time since exiting this station in the preceding cycle. The intervisit time can be divided into the sum of the switchover times and the time which the server spends at station 2 between two consecutive visits at station 1. This latter time can be split in two parts. First, the server stays the minimum sojourn time T_2. The second part consists of the time which the server takes to process the possibly remaining messages. This part depends on the number of messages present at time T_2. Given that the server finds k messages upon arrival at station 2, there are l messages present with probability $P_{k,l}(T_2)$ after spending the minimum sojourn time. Then, the length of the second part has the density $g_2^{(*l)}$, which denotes the density of the sum of l independent busy periods at station 2. We recall that the arrival process at station 1 is a Poisson process with arrival rate λ_1. The probability of finding n messages at station 1 is given by a Poisson distribution with parameter $\lambda_1 t$ if the intervisit time of the server equals t. Therefore, we can deduce the equation

$$\pi_n^{(1)} = \sum_{k=0}^{\infty} \pi_k^{(2)} \sum_{l=0}^{\infty} P_{k,l}(T_2) \int_0^{\infty} e^{-\lambda_1(r_0+T_2+x)} \frac{(\lambda_1(r_0+T_2+x))^n}{n!} g_2^{(*l)}(x) \mathrm{d}x$$

for deterministic switchover times and get the coefficients for an infinite linear system of equations $\pi^{(1)} = A\pi^{(2)}$. In the same manner as above, there is a system $\pi^{(2)} = B\pi^{(1)}$.

If the switchover times are non-deterministic, we cannot proceed in such a straightforward way. Instead, we study the queue length distribution at server departure instants. We note that the queue at departure instants is always empty at the current station. We denote by $\nu_n^{(i)}$ the steady-state probability that there are n messages waiting at the other station upon exit from station i. In the following, we provide an explanation for the equation

$$\begin{aligned} \nu_n^{(1)} = \sum_{k=0}^{\infty} \nu_k^{(2)} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \Bigg(\int_0^{\infty} e^{-(\lambda_1+\lambda_2)x} \frac{(\lambda_1 x)^m}{m!} \frac{(\lambda_2 x)^j}{j!} \mathrm{d}F_{R_2}(x) \\ \times \sum_{l=0}^{\infty} P_{k+m,l}(T_1) \int_0^{\infty} e^{-\lambda_2(T_1+x)} \frac{(\lambda_2(T_1+x))^{n-j}}{(n-j)!} g_1^{(*l)}(x) \mathrm{d}x \Bigg), \end{aligned} \tag{1.12}$$

which consists of similar terms as above. Given that there are k messages waiting at station 1 upon exit from station 2, we have m message arrivals at station 1 and j message arrivals at station 2 while switching to station 1. Therefore, there are $k+m$ messages waiting at station 1 upon arrival at this station. In order to obtain a queue length of n messages at station 2 upon exit from station 1, a total of $n-j$ messages have to arrive at station 2 during this stay. Then, equation (1.12) follows by considering all possible configurations of indices.

From (1.12) and the corresponding observation, we get two systems of equations $\nu^{(1)} = \widetilde{A}\nu^{(2)}$ and $\nu^{(2)} = \widetilde{B}\nu^{(1)}$, where the coefficients of $\widetilde{A}$ are given in (1.12). Finally, we are able to determine $\pi_n^{(i)}$ by

$$\pi_n^{(1)} = \sum_{k=0}^{n} \nu_k^{(2)} \int_0^{\infty} e^{-\lambda_1 x} \frac{(\lambda_1 x)^{n-k}}{(n-k)!} \mathrm{d}F_{R_2}(x). \tag{1.13}$$

For $\pi^{(2)}$, we have to reverse the roles of 1 and 2.

The expected sojourn time c_i. Using the solution $\pi^{(i)}$, we obtain the expected sojourn time

$$c_i = T_i + \sum_{k=0}^{\infty} \pi_k^{(i)} \sum_{l=0}^{\infty} l P_{k,l}(T_i) \frac{b_i}{1-\rho_i}$$

which the server spends at station i per cycle. Here, the series

$$\sum_{l=0}^{\infty} l P_{k,l}(T_i)$$

is the expectation of the number of messages present at station i after spending the minimum sojourn time T_i given that there are k messages present upon arrival of the server. As already mentioned above, the quotient $b_i/(1-\rho_i)$ is the expected length of a busy period.

The expected backward recurrence time $\boldsymbol{w_i}$. In order to determine the quantity w_i, we recall the condition that a point in time while the server is waiting is randomly chosen. We distinguish how many messages are waiting upon arrival of the server at the station. Therefore, we obtain

$$w_i = \sum_{k=0}^{\infty} p_k^{(i)} \frac{\int_0^{T_i} x P_{k,0}(x)\mathrm{d}x}{\int_0^{T_i} P_{k,0}(x)\mathrm{d}x},$$

where $p_k^{(i)}$ denotes the probability of choosing a waiting period during a stay with k messages waiting upon arrival of the server. Similar to Strategy III above, the quotient is the expectation of the elapsed time since arriving at station i at a random point in time while waiting at station i given that there are k messages waiting upon arrival of the server.

It remains to determine the coefficient $p_k^{(i)}$. The basic observation is that $p_k^{(i)}$ is proportional to the product of the probability $\pi_k^{(i)}$ that the server finds k messages waiting upon arrival at station i and the expected length of the total waiting time during the stay at such a station. This last-mentioned expected length equals $\int_0^{T_i} P_{k,0}(x)\mathrm{d}x$ because, given that the server finds k messages waiting upon arrival, we have

$$\begin{aligned}
&\mathbb{E}[\text{total waiting time during the stay}] \\
&= \int \int_0^{T_i} \mathbb{1}_{\{\text{server is waiting at time x}\}} \mathrm{d}x \mathrm{d}\mathbb{P} \\
&= \int_0^{T_i} \int \mathbb{1}_{\{\text{server is waiting at time x}\}} \mathrm{d}\mathbb{P} \mathrm{d}x \\
&= \int_0^{T_i} P_{k,0}(x)\mathrm{d}x.
\end{aligned}$$

Hence, the probability $p_k^{(i)}$ is given by

$$p_k^{(i)} = \frac{\pi_k^{(i)} \int_0^{T_i} P_{k,0}(x)\mathrm{d}x}{\sum_{l=0}^{\infty} \pi_l^{(i)} \int_0^{T_i} P_{l,0}(x)\mathrm{d}x}. \tag{1.14}$$

The conditional mean switchover time $\widetilde{r}_i$. If the switchover time from station i to the next station is deterministic, we get $\widetilde{r}_i = r_i$ (cf. Remark 7). Otherwise, the conditional mean switchover time $\widetilde{r}_i$ from station i to the next station, given a random point in time while waiting at station $i+1$, can be determined as follows: We restrict the computation to $i = 2$ for the sake of clarity. First, we introduce the events

$$\begin{aligned} A_l &:= \{\text{there are } l \text{ messages waiting at station 1 upon exit from station 2}\}, \\ B_j &:= \{\text{there are } j \text{ messages arriving at 1 while switching from 2 to 1}\}, \\ C_k &:= \{\text{there are } k \text{ messages waiting at station 1 upon arrival}\} \end{aligned}$$

for all $j, k, l \in \mathbb{N}_0$. We distinguish how many messages are waiting upon arrival of the server at station 1 just like above. We get

$$\widetilde{r}_2 = \sum_{k=0}^{\infty} p_k^{(1)} \, \mathbb{E}[R_2 \mid C_k], \tag{1.15}$$

where $p_k^{(i)}$ is given in (1.14). Now, we are left with the specification of the quantity $\mathbb{E}[R_2 \mid C_k]$. We make use of

$$C_k = \bigcup_{j=0}^{k} A_{k-j} \cap B_j$$

and obtain

$$\mathbb{E}[R_2 \mid C_k] = \sum_{j=0}^{k} \frac{\mathbb{P}(A_{k-j} \cap B_j)}{\mathbb{P}(C_k)} \, \mathbb{E}[R_2 \mid A_{k-j} \cap B_j].$$

Due to the independence of the events A_{k-j} and B_j, and the fact that A_{k-j} does not influence the switchover time R_2, we get

$$\mathbb{E}[R_2 \mid C_k] = \sum_{j=0}^{k} \frac{\mathbb{P}(A_{k-j})\mathbb{P}(B_j)}{\mathbb{P}(C_k)} \, \mathbb{E}[R_2 \mid B_j]. \tag{1.16}$$

It remains to determine the quantities on the right hand side of (1.16). We can represent the event B_j as

$$B_j = \left\{ \sum_{l=1}^{j} e_l \leq R < \sum_{l=1}^{j+1} e_l \right\}, \tag{1.17}$$

where $(e_l)_l$ is a sequence of independent and exponentially distributed random variables with parameter 1 which are independent of $R := \lambda_1 R_2$ as well. We get

$$\lambda_1 \mathbb{E}\left[R_2 \mid B_j\right] = \frac{\mathbb{E}\left[R \, \mathbb{1}_{B_j}\right]}{\mathbb{E}\left[\mathbb{1}_{B_j}\right]} = \frac{\mathbb{E}_R\left[R \, \mathbb{E}_{(e_l)_l}\left[\mathbb{1}_{B_j}\right]\right]}{\mathbb{E}_R\left[\mathbb{E}_{(e_l)_l}\left[\mathbb{1}_{B_j}\right]\right]}.$$

We use the fact that the sum of independent and identically exponentially distributed random variables is Erlang distributed. We thus compute

$$
\begin{aligned}
\mathbb{E}_{(e_l)_l}\left[\mathbb{1}_{B_j}\right] &= \int_0^R \int_{R-y}^{\infty} \mathrm{d}\mathbb{P}_{e_{j+1}}(t)\mathrm{d}\mathbb{P}_{\sum_{l=1}^{j} e_l}(y) \\
&= \int_0^R \int_{R-y}^{\infty} e^{-t}\mathrm{d}t\, \frac{y^{j-1}}{(j-1)!}e^{-y}\mathrm{d}y \\
&= \int_0^R e^{-(R-y)}\frac{y^{j-1}}{(j-1)!}e^{-y}\mathrm{d}y \\
&= \frac{e^{-R}}{(j-1)!}\int_0^R y^{j-1}\mathrm{d}y \\
&= \frac{R^j}{j!}e^{-R}.
\end{aligned}
$$

Therefore, we obtain

$$
\mathbb{E}[R_2 \mid B_j] = \frac{\mathbb{E}_R\left[R^{j+1}e^{-R}\right]}{\lambda_1\mathbb{E}_R\left[R^j e^{-R}\right]} = \frac{\int_0^\infty x e^{-\lambda_1 x}\frac{(\lambda_1 x)^j}{j!}\mathrm{d}F_{R_2}(x)}{\int_0^\infty e^{-\lambda_1 x}\frac{(\lambda_1 x)^j}{j!}\mathrm{d}F_{R_2}(x)} \tag{1.18}
$$

and

$$
\mathbb{P}(B_j) = \mathbb{E}\left[\mathbb{1}_{B_j}\right] = \int_0^\infty e^{-\lambda_1 x}\frac{(\lambda_1 x)^j}{j!}\mathrm{d}F_{R_2}(x).
$$

Finally, we observe

$$
\mathbb{P}(C_k) = \sum_{j=0}^{k}\mathbb{P}(A_{k-j})\mathbb{P}(B_j)
$$

due to the independence and recall $\mathbb{P}(A_{k-j}) = \nu_{k-j}^{(2)}$.

Strategy IV

As above, $\pi_n^{(i)}$ is the steady-state probability that the server finds n messages waiting upon arrival at station i. The method that we use to establish the characterising system coincides with the method for Strategy II. The probability $\pi_n^{(1)}$ depends on the intervisit time of the server, which consists of the switchover times and the time that the server spends at station 2 between two consecutive visits at station 1.

We have to distinguish whether there is no message or at least one message waiting at station 2 because it influences the activation of the timer. In the first case, either a new message arrives before the timer expires and a busy period starts, or there is no message arrival and the server waits the whole time T_2. For deterministic switchover times, we obtain

$$\pi_n^{(1)} = \pi_0^{(2)} \left(\int_0^{T_2} \int_0^{\infty} e^{-\lambda_1(r_0+x+y)} \frac{(\lambda_1(r_0+x+y))^n}{n!} g_2(x)\mathrm{d}x \, \lambda_2 e^{-\lambda_2 y}\mathrm{d}y \right.$$
$$\left. + \, e^{-\lambda_1(r_0+T_2)} \frac{(\lambda_1(r_0+T_2))^n}{n!} e^{-\lambda_2 T_2} \right)$$
$$+ \sum_{k=1}^{\infty} \pi_k^{(2)} \int_0^{\infty} e^{-\lambda_1(r_0+x)} \frac{(\lambda_1(r_0+x))^n}{n!} g_2^{(*k)}(x)\mathrm{d}x.$$

Once again, we get systems of equations $\pi^{(1)} = A\pi^{(2)}$ and $\pi^{(2)} = B\pi^{(1)}$. We note that we are only interested in $\pi_0^{(i)}$ in the end.

In the case of non-deterministic switchover times, we study the steady-state probability $\nu_n^{(i)}$ that there are n messages waiting at the other station upon exit from station i. We obtain

$$\nu_n^{(1)} = \nu_0^{(2)} \sum_{j=0}^{n} \left(\int_0^{\infty} e^{-(\lambda_1+\lambda_2)x} \frac{(\lambda_1 x)^0}{0!} \frac{(\lambda_2 x)^j}{j!} \mathrm{d}F_{R_2}(x) \right.$$
$$\times \left(\int_0^{T_1} \int_0^{\infty} e^{-\lambda_2(x+y)} \frac{(\lambda_2(x+y))^{n-j}}{(n-j)!} g_1(x)\mathrm{d}x \, \lambda_1 e^{-\lambda_1 y}\mathrm{d}y \right.$$
$$\left.\left. + \, e^{-\lambda_2 T_1} \frac{(\lambda_2 T_1)^{n-j}}{(n-j)!} e^{-\lambda_1 T_1} \right)\right)$$
$$+ \sum_{k=0}^{\infty} \nu_k^{(2)} \sum_{\substack{m=0 \\ m+k\neq 0}}^{\infty} \sum_{j=0}^{n} \left(\int_0^{\infty} e^{-(\lambda_1+\lambda_2)x} \frac{(\lambda_1 x)^m}{m!} \frac{(\lambda_2 x)^j}{j!} \mathrm{d}F_{R_2}(x) \right.$$
$$\left. \times \int_0^{\infty} e^{-\lambda_2 x} \frac{(\lambda_2 x)^{n-j}}{(n-j)!} g_1^{(*(k+m))}(x)\mathrm{d}x \right)$$

and get two systems of equations $\nu^{(1)} = \widetilde{A}\nu^{(2)}$ and $\nu^{(2)} = \widetilde{B}\nu^{(1)}$. Finally, we can compute $\pi_n^{(i)}$ exactly as mentioned in (1.13) for Strategy II.

The expected waiting time f_i. Let E_i be an exponentially distributed random variable with parameter λ_i, which represents the interarrival time of messages at station i. Then, $\min(E_i, T_i)$ is the random length of a waiting period at station i. The timer at station i is activated if and only if the server finds this station empty upon arrival. Therefore, we can conclude

$$f_i = \pi_0^{(i)} \mathbb{E}\left[\min(E_i, T_i)\right] = \frac{\pi_0^{(i)}}{\lambda_i} \left(1 - e^{-\lambda_i T_i}\right)$$

for the expected waiting time at station i per cycle because we have

$$
\begin{aligned}
\mathbb{E}\left[\min(E_i, T_i)\right] &= \int_0^{T_i} x\lambda_i e^{-\lambda_i x} \mathrm{d}x + T_i \mathbb{P}(E_i > T_i) \\
&= - T_i e^{-\lambda_i T_i} + \int_0^{T_i} e^{-\lambda_i x} \mathrm{d}x + T_i e^{-\lambda_i T_i} \qquad (1.19) \\
&= \frac{1 - e^{-\lambda_i T_i}}{\lambda_i},
\end{aligned}
$$

where we used integration by parts in (1.19).

The expected backward recurrence time w_i. The quantity w_i equals the expected residual time of a waiting period, which is given by

$$
\frac{\mathbb{E}\left[\min(E_i, T_i)^2\right]}{2\mathbb{E}\left[\min(E_i, T_i)\right]}
$$

(cf. (5.15) in [25]). A short calculation shows

$$
\begin{aligned}
\mathbb{E}\left[\min(E_i, T_i)^2\right] &= \int_0^{T_i} x^2 \lambda_i e^{-\lambda_i x} \mathrm{d}x + T_i^2 \mathbb{P}(E_i > T_i) \\
&= - T_i^2 e^{-\lambda_i T_i} + \frac{2}{\lambda_i} \int_0^{T_i} x\lambda_i e^{-\lambda_i x} \mathrm{d}x + T_i^2 e^{-\lambda_i T_i} \qquad (1.20) \\
&= \frac{2}{\lambda_i} \left(\frac{1 - e^{-\lambda_i T_i}}{\lambda_i} - T_i e^{-\lambda_i T_i} \right),
\end{aligned}
$$

where we used integration by parts in (1.20) and we note that the integral in (1.20) has already appeared in the computation of $\mathbb{E}\left[\min(E_i, T_i)\right]$ in the preceding paragraph. Finally, we can deduce

$$
w_i = \frac{\mathbb{E}\left[\min(E_i, T_i)^2\right]}{2\mathbb{E}\left[\min(E_i, T_i)\right]} = \frac{1}{\lambda_i} - \frac{T_i}{e^{\lambda_i T_i} - 1}.
$$

The conditional mean switchover time $\widetilde{r}_i$. If the switchover time is deterministic, we just have $\widetilde{r}_i = r_i$ (cf. Remark 7). Now, we consider a non-deterministic switchover time. Similar but easier than for Strategy II, the quantity $\widetilde{r}_2$ is just the mean switchover time given that there is no arrival at station 1 while switching to this station. We get

$$
\widetilde{r}_2 = \mathbb{E}[R_2 \mid B_0] = \frac{\int_0^\infty x e^{-\lambda_1 x} \mathrm{d}F_{R_2}(x)}{\int_0^\infty e^{-\lambda_1 x} \mathrm{d}F_{R_2}(x)} \qquad (1.21)
$$

due to (1.18) and we can represent $\widetilde{r}_1$ in an analogous manner.

1.3.3 Proofs of the 'worth-waiting' results

First, we state two facts that we use later to prove that it is worth waiting with Strategy II if it is worth waiting with Strategy IV. Lemma 9 concerns an estimate for the mean switchover time given a certain number of message arrivals while switching.

Lemma 9. *There is a constant $C > 0$ such that*

$$\mathbb{E}[R_2 \mid B_j] \leq C\left(j^2+1\right)$$

for all $j \in \mathbb{N}$ with the notation from (1.17)*.*

Proof: We recall

$$\mathbb{E}[R_2 \mid B_j] = \frac{\mathbb{E}_R\left[R^{j+1}e^{-R}\right]}{\lambda_1 \mathbb{E}_R\left[R^j e^{-R}\right]}$$

for $R := \lambda_1 R_2$ from (1.18) and we introduce the non-negative random variable X by

$$\mathbb{E}\left[f(X)\right] := \frac{\mathbb{E}_R\left[f(R)e^{-R}\right]}{\mathbb{E}_R\left[e^{-R}\right]}, \quad \text{for } f \in C_b.$$

Hence, we have to show that there is a $C > 0$ such that

$$\frac{\mathbb{E}\left[X^{j+1}\right]}{\mathbb{E}\left[X^j\right]} \leq C\lambda_1\left(j^2+1\right)$$

for all $j \in \mathbb{N}$.

We consider the first case that $X \leq 1$ holds almost surely. Then, we have $\mathbb{E}[X^{j+1}] \leq \mathbb{E}[X^j]$ that implies

$$\frac{\mathbb{E}\left[X^{j+1}\right]}{\mathbb{E}\left[X^j\right]} \leq 1.$$

Next, we continue with the other case $\mathbb{P}(X > 1) > 0$ and get

$$\begin{aligned}
\frac{\mathbb{E}\left[X^{j+1}\right]}{\mathbb{E}[X^j]} &= \frac{\mathbb{E}\left[X^{j+1}\mathbb{1}_{\{X>1\}}\right] + \mathbb{E}\left[X^{j+1}\mathbb{1}_{\{X\leq 1\}}\right]}{\mathbb{E}\left[X^{j}\mathbb{1}_{\{X>1\}}\right] + \mathbb{E}\left[X^{j}\mathbb{1}_{\{X\leq 1\}}\right]} \\
&\leq \frac{\mathbb{E}\left[X^{j+1}\mathbb{1}_{\{X>1\}}\right] + 1}{\mathbb{E}\left[X^{j}\mathbb{1}_{\{X>1\}}\right] + 0} \\
&\leq \frac{\mathbb{E}\left[X^{j+1}\mathbb{1}_{\{X>1\}}\right]}{\mathbb{E}\left[X^{j}\mathbb{1}_{\{X>1\}}\right]} + \frac{1}{\mathbb{P}(X>1)}.
\end{aligned}$$

We define $Y := X\mathbb{1}_{\{X>1\}}$ and use Hölder's inequality

$$\mathbb{E}\left[Y^{j+1}\right] = \mathbb{E}\left[Y^{j-1}Y^2\right] \leq \left(\mathbb{E}\left[Y^{(j-1)p}\right]\right)^{1/p}\left(\mathbb{E}\left[Y^{2q}\right]\right)^{1/q}$$

with $p \coloneqq j/j-1$ and $q \coloneqq j$ in order to obtain

$$\frac{\mathbb{E}\left[X^{j+1}\mathbb{1}_{\{X>1\}}\right]}{\mathbb{E}\left[X^{j}\mathbb{1}_{\{X>1\}}\right]} = \frac{\mathbb{E}\left[Y^{j+1}\right]}{\mathbb{E}[Y^{j}]} \leq \left(\mathbb{E}\left[Y^{j}\right]\right)^{-1/j}\left(\mathbb{E}\left[Y^{2j}\right]\right)^{1/j}. \tag{1.22}$$

Furthermore, we can conclude

$$\mathbb{E}\left[Y^{j}\right] \leq \left(\mathbb{E}\left[Y^{j+1}\right]\right)^{j/(j+1)}$$

from Hölder's inequality. We observe that $\left(\mathbb{E}\left[Y^{j}\right]\right)^{1/j}$ is monotonically increasing in j. Hence, $\left(\mathbb{E}\left[Y^{j}\right]\right)^{-1/j}$ is bounded above by $(\mathbb{E}[Y])^{-1}$. In combination with (1.22), it remains to deal with $\mathbb{E}\left[Y^{2j}\right]$. We can bound this quantity by

$$\begin{aligned}\mathbb{E}\left[Y^{2j}\right] &= \int_0^\infty 2jt^{2j-1}\mathbb{P}(Y>t)\mathrm{d}t\\ &\leq 2j\int_0^\infty t^{2j-1}\beta e^{-t}\mathrm{d}t\\ &= \beta 2j\Gamma(2j) = \beta(2j)! \leq \beta(2j)^{2j},\end{aligned}$$

where we used an estimate of the tail (see below) and the definition of the gamma function. We have

$$\mathbb{P}(Y>t) = \mathbb{P}(X\mathbb{1}_{\{X>1\}}>t) \leq \mathbb{P}(X>t)$$

and

$$\mathbb{P}(X>t) = 1 - \frac{\mathbb{E}_R\left[\mathbb{1}_{\{R\leq t\}}e^{-R}\right]}{\mathbb{E}_R\left[e^{-R}\right]} = \frac{\mathbb{E}_R\left[\mathbb{1}_{\{R>t\}}e^{-R}\right]}{\mathbb{E}_R\left[e^{-R}\right]} \leq \beta e^{-t}$$

with $\beta \coloneqq \left(\mathbb{E}_R\left[e^{-R}\right]\right)^{-1} \geq 1$. Combining all the relevant inequalities, we see

$$\frac{\mathbb{E}\left[X^{j+1}\right]}{\mathbb{E}[X^{j}]} \leq (\mathbb{E}[Y])^{-1}\beta(2j)^2 + \frac{1}{\mathbb{P}(X>1)}$$

for this latter case and this finishes the proof by setting C appropriately. □

Lemma 10 contains the fact that if there may be an additional waiting time due to a greater wait-and-see parameter $\widetilde{T}_1 \geq T_1$, rather more messages arrive per cycle. Therefore, the probability of finding an empty queue upon arrival at station 1 becomes smaller.

Lemma 10. *We consider a polling model with Strategy II, $N=2$ stations and $T_2=0$. Given a $\bar{T}_1>0$, we have*

$$\pi_{\inf}(\bar{T}_1) \coloneqq \inf_{T_1\in[0,\bar{T}_1]} \pi_0^{(1)}(T_1) = \pi_0^{(1)}(\bar{T}_1) > 0.$$

Proof: The idea of the proof is as follows: We can construct an appropriate coupling of two processes representing the polling models with wait-and-see parameters T_1 and $\widetilde{T}_1$ for $0 \leq T_1 \leq \widetilde{T}_1$. Due to the construction, the queue length at station 1 upon exit from station 2 is always greater for the process with $\widetilde{T}_1$ instead of T_1. Combining this observation and the ergodic theorem for Markov chains, we obtain

$$\nu_0^{(2)}(T_1) \geq \nu_0^{(2)}(\widetilde{T}_1).$$

This inequality is equivalent to

$$\pi_0^{(1)}(T_1) \geq \pi_0^{(1)}(\widetilde{T}_1)$$

due to (1.13). Then, we can conclude that $\pi_{\inf}(\bar{T}_1) = \pi_0^{(1)}(\bar{T}_1) > 0$.

We denote by $(Z_n)_n$ and $(\widetilde{Z}_n)_n$ the queue length process at station 1 upon exit from station 2 for the polling model with wait-and-see parameter T_1 and $\widetilde{T}_1$, respectively. The index $n \in \mathbb{N}$ denotes the cycle number and we set $Z_0 = \widetilde{Z}_0 = 0$. Both Markov chains are coupled as follows: We consider the sample space $\Omega := \Omega_0 \times \widetilde{\Omega} \times \Omega_R$. The switchover times are encoded in Ω_R, the arrival and service times for both stations in Ω_0, and $\widetilde{\Omega}$ is a copy of Ω_0. The value of $Z_n(\omega)$ for $\omega = (\omega_0, \widetilde{\omega}, \omega_R)$ is totally determined by ω_0 and ω_R. Mainly, the evolution of the process $\widetilde{Z}_n$ is determined by ω_0 and ω_R just like for Z_n but there is one exception: If the sojourn time at a station is greater than the corresponding one (with parameter T_1), we use the arrival and service times from $\widetilde{\omega}$ for the additional sojourn time. This situation occurs if and only if exiting station 1 for the process with T_1 happens before $\widetilde{T}_1$ or messages, which have arrived during an extended sojourn time at the preceding station, have still to be processed at the point in time (in the process with T_1) of exiting the current station. We further declare that at each station the processing of messages that are only present because of $\widetilde{T}_1 > T_1$ compared to the process with T_1 does not take place until all other messages at this station have been processed.

Due to this construction, we can deduce $Z_n \leq \widetilde{Z}_n$ for all $n \in \mathbb{N}$ because a possibly greater cycle time cannot result in a smaller queue length (it rather results in a greater queue length). Therefore, we obtain $\mathbb{1}_{\{Z_n=0\}} \geq \mathbb{1}_{\{\widetilde{Z}_n=0\}}$ for all $n \in \mathbb{N}$. Using the ergodic theorem for Markov chains, we get

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{1}_{\{Z_k=0\}} = \nu_0^{(2)}(T_1)\right) = 1,$$

where the same holds for $\widetilde{Z}_k$ and $\widetilde{T}_1$ instead of Z_k and T_1. Combining this equation with the observation above, we obtain $\nu_0^{(2)}(T_1) \geq \nu_0^{(2)}(\widetilde{T}_1)$ and the lemma follows. □

In the following, we make use of Theorems 1 and 2 in order to prove the 'worth-waiting' results. For the purpose of comparison, we recall the formula

$$\bar{D}^{\text{exh}} = \frac{\sum_{i=1}^{2} \lambda_i b_i^{(2)}}{2(1-\rho_0)} + \frac{r_0 \rho_1 \rho_2}{\rho_0(1-\rho_0)} + \frac{r_0^{(2)}}{2r_0}$$

for the mean average queueing delay in a polling model with the exhaustive strategy from (1.2) by setting $f_1 = f_2 = 0$. Hence, we can rearrange the formula for the delay into $\bar{D} = \bar{D}^{\text{exh}} + \Delta\bar{D}$ with

$$\begin{aligned} \Delta\bar{D} \coloneqq &- \frac{r_0^{(2)}}{2r_0} + \frac{r_0^{(2)}}{2(r_0+f_0)} \\ &+ \frac{\rho_2 f_1}{\rho_0(r_0+f_0)}(r_1 + \widetilde{r}_2 + w_1) \\ &+ \frac{\rho_1 f_2}{\rho_0(r_0+f_0)}(\widetilde{r}_1 + r_2 + w_2), \end{aligned} \tag{1.23}$$

where $\widetilde{r}_i = r_i$ holds for $i = 1, 2$ in the case of Strategy III (cf. Remark 7).

Proof of Theorem 3: Due to $T_2 = 0$, we have $f_2 = 0$ and the term in (1.23) vanishes. It is worth waiting at station 1 if and only if there is a positive parameter of the wait-and-see strategy such that $\Delta\bar{D} < 0$. Since the expected waiting time at station 1 equals the total expected waiting time per cycle ($f_1 = f_0$), we rearrange inequality $\Delta\bar{D} < 0$ into

$$\frac{1}{r_0+f_1}\left(\frac{r_0^{(2)}}{2} + \frac{\rho_2}{\rho_0} f_1 \left(r_1 + \widetilde{r}_2 + w_1\right)\right) < \frac{r_0^{(2)}}{2r_0},$$

whose validity is equivalent to

$$\left(-\frac{r_0^{(2)}}{2r_0} + \frac{\rho_2}{\rho_0}\left(r_1 + \widetilde{r}_2 + w_1\right)\right) f_1 < 0. \tag{1.24}$$

We recall that w_i and f_i are non-negative quantities. Moreover, we observe that $f_i > 0$ holds for all $T_i > 0$. This can be seen by using the expected sojourn times for Strategy III and by using the steady-state probabilities for Strategies II and IV. From now on, we treat Strategies II–IV separately.

Strategy III. According to Remark 7, we have $r_1 + \widetilde{r}_2 = r_0$. For all $T_1 > 0$, we see from (1.11) that w_1 is greater than the expected length of the first busy period at station 1, i.e., there is a function $\Delta_1(T_1) > 0$ such that

$$w_1 = (r_0 + c_2)\frac{\rho_1}{1-\rho_1} + \Delta_1(T_1).$$

We insert this representation of w_1 into (1.24), make use of (1.10) and obtain that (1.24) is equivalent to

$$-\frac{r_0^{(2)}}{2r_0}+\frac{\rho_2}{\rho_0}\left(\frac{1-\rho_2}{1-\rho_0}r_0+\frac{\rho_1\rho_2}{1-\rho_0}\left(T_1+\frac{q_1(T_1)b_1}{1-\rho_1}\right)+\Delta_1(T_1)\right)<0. \quad (1.25)$$

Because of the property that both functions $\Delta_1(T_1)$ and $q_1(T_1)$ converge to zero for $T_1\to 0$, we get the sufficient condition

$$\frac{r_0^{(2)}}{2r_0^2}-\frac{\rho_2(1-\rho_2)}{\rho_0(1-\rho_0)}>0 \quad (1.26)$$

for 'it is worth waiting at station 1'. In order to establish the necessity of this condition, we proceed in the following way: If we assume that (1.26) does not hold, inequality (1.25) is not satisfied for all $T_1>0$ because $\Delta_1(T_1)$ and $q_1(T_1)$ are non-negative, and we see that it is not worth waiting at station 1.

Strategy IV. The difference to Strategy III is the fact that w_1 does not have to be greater than the expected length of the first busy period at station 1. We look at

$$-\frac{r_0^{(2)}}{2r_0}+\frac{\rho_2}{\rho_0}\left(r_1+\widetilde{r}_2+w_1\right)<0 \quad (1.27)$$

from (1.24) and observe $w_1\le T_1$ because a waiting period ends at the latest when the timer expires. In the same manner as above, we deduce the necessary and sufficient condition

$$\frac{r_0^{(2)}}{2r_0\left(r_1+\widetilde{r}_2^{\mathrm{IV}}\right)}-\frac{\rho_2}{\rho_0}>0$$

for 'it is worth waiting at station 1' with $\widetilde{r}_2^{\mathrm{IV}}$ given in (1.21). In the case of deterministic switchover times, we just replace $r_0^{(2)}$ and $\widetilde{r}_2^{\mathrm{IV}}$ by r_0^2 and r_2, respectively.

Strategy II. We look again at (1.27) as with Strategy IV and note that $w_1\le T_1$ holds since waiting periods can only happen within the minimum sojourn time T_1. In contrast to Strategy IV, the conditional mean switchover time $\widetilde{r}_2^{\mathrm{II}}$ depends on the parameter T_1.

First, we prove that it is worth waiting with Strategy IV if it is worth waiting with Strategy II. Therefore, we assume that there is a $T_1>0$ such that (1.27) holds for Strategy II. We have to conclude that (1.4) is satisfied, which can be seen easily if we have $\widetilde{r}_2^{\mathrm{IV}}\le\widetilde{r}_2^{\mathrm{II}}(T_1)$ for all $T_1>0$. We continue with proving this inequality. We recall

$$\widetilde{r}_2^{\mathrm{IV}}=\mathbb{E}[R_2\mid B_0]$$

from (1.21) and

$$\widetilde{r}_2^{\mathrm{II}} = \sum_{k=0}^{\infty} p_k^{(1)} \sum_{j=0}^{k} \frac{\mathbb{P}(A_{k-j})\mathbb{P}(B_j)}{\mathbb{P}(C_k)} \mathbb{E}[R_2 \mid B_j]$$

from (1.15) and (1.16). We define the random variable $R := \lambda_1 R_2$ and obtain due to the Cauchy-Schwarz inequality

$$\mathbb{E}\left[R^{j+1}e^{-R}\right]^2 = \mathbb{E}\left[\left(R^j e^{-R}\right)^{1/2}\left(R^{j+2}e^{-R}\right)^{1/2}\right]^2 \leq \mathbb{E}\left[R^j e^{-R}\right]\mathbb{E}\left[R^{j+2}e^{-R}\right]$$

for all $j \in \mathbb{N}_0$. We use this inequality and the representation of $\mathbb{E}[R_2 \mid B_j]$ in (1.18) to get

$$\mathbb{E}[R_2 \mid B_j] = \frac{\mathbb{E}\left[R^{j+1}e^{-R}\right]}{\lambda_1 \mathbb{E}\left[R^j e^{-R}\right]} \leq \frac{\mathbb{E}\left[R^{j+2}e^{-R}\right]}{\lambda_1 \mathbb{E}\left[R^{j+1}e^{-R}\right]} = \mathbb{E}[R_2 \mid B_{j+1}].$$

It follows that $\mathbb{E}[R_2 \mid B_0] \leq \mathbb{E}[R_2 \mid B_j]$ for all $j \in \mathbb{N}$. This fact suffices in order to deduce $\widetilde{r}_2^{\mathrm{IV}} \leq \widetilde{r}_2^{\mathrm{II}}(T_1)$ for all $T_1 > 0$.

Next, we have to prove that it is worth waiting with Strategy II if it is worth waiting with Strategy IV. Let (1.4) be satisfied, i.e., there is a $T_1^{\mathrm{IV}} > 0$ such that (1.27) holds for $\widetilde{r}_2^{\mathrm{IV}}$ and $w_1^{\mathrm{IV}}(T_1^{\mathrm{IV}})$. We are done if there is a $T_1 > 0$ such that

$$\widetilde{r}_2^{\mathrm{II}}(T_1) + w_1^{\mathrm{II}}(T_1) \leq \widetilde{r}_2^{\mathrm{IV}} + w_1^{\mathrm{IV}}(T_1^{\mathrm{IV}})$$

because (1.27) is the condition for 'it is worth waiting with Strategy II' as well. We observe

$$\begin{aligned}
\widetilde{r}_2^{\mathrm{II}} &= p_0^{(1)}\mathbb{E}[R_2 \mid B_0] + \sum_{k=1}^{\infty} p_k^{(1)} \sum_{j=0}^{k} \frac{\mathbb{P}(A_{k-j})\mathbb{P}(B_j)}{\mathbb{P}(C_k)} \mathbb{E}[R_2 \mid B_j] \\
&\leq \mathbb{E}[R_2 \mid B_0] + \sum_{k=1}^{\infty} p_k^{(1)}\mathbb{E}[R_2 \mid B_k]
\end{aligned}$$

and define $\varepsilon := w_1^{\mathrm{IV}}(T_1^{\mathrm{IV}})/2$. Due to $\widetilde{r}_2^{\mathrm{IV}} = \mathbb{E}[R_2 \mid B_0]$ and $w_1^{\mathrm{II}}(T_1) \leq T_1$, it suffices to show that there is a positive $T_1 < \varepsilon$ such that

$$\sum_{k=1}^{\infty} p_k^{(1)}\mathbb{E}[R_2 \mid B_k] < \varepsilon.$$

We recall

$$p_k^{(1)} = \frac{\pi_k^{(1)} \int_0^{T_1} P_{k,0}(x)\mathrm{d}x}{\sum_{l=0}^{\infty} \pi_l^{(1)} \int_0^{T_1} P_{l,0}(x)\mathrm{d}x}$$

from (1.14). First, we estimate the quantity $\int_0^{T_1} P_{k,0}(x)\mathrm{d}x$, which is the expected length of the total waiting time during the stay at station 1 given that there are k messages waiting upon arrival. We get

$$\begin{aligned}\int_0^{T_1} P_{0,0}(x)\mathrm{d}x \geq& T_1\, \mathbb{P}(\text{no message arrives at station 1 within the time } T_1)\\ =& T_1 e^{-\lambda_1 T_1}\end{aligned}$$

and

$$\begin{aligned}\int_0^{T_1} P_{k,0}(x)\mathrm{d}x \leq& T_1\, \mathbb{P}(\text{the length of the first busy period} \leq T_1)\\ \leq& T_1\, \mathbb{P}(\text{the sum of } k \text{ independent service times} \leq T_1)\\ \leq& T_1 \left(1 - e^{-\mu_1 T_1} \sum_{j=0}^{k-1} \frac{(\mu_1 T_1)^j}{j!}\right) \qquad (1.28)\\ =& T_1 e^{-\mu_1 T_1} \left(e^{\mu_1 T_1} - \sum_{j=0}^{k-1} \frac{(\mu_1 T_1)^j}{j!}\right)\\ =& T_1 e^{-\mu_1 T_1} \sum_{j=k}^{\infty} \frac{(\mu_1 T_1)^j}{j!}\\ =& T_1 e^{-\mu_1 T_1} (\mu_1 T_1)^k \sum_{j=0}^{\infty} \frac{(\mu_1 T_1)^j}{(j+k)\cdots(j+1)j!}\\ \leq& T_1 (\mu_1 T_1)^k\end{aligned}$$

for all $k \in \mathbb{N}$, where we used the Erlang(k, μ_1) distribution function in (1.28). Therefore, we can bound $p_k^{(1)}$ for all $k \in \mathbb{N}$ from above by

$$p_k^{(1)} \leq \frac{T_1(\mu_1 T_1)^k}{\pi_0^{(1)} T_1 e^{-\lambda_1 T_1}} = \frac{e^{\lambda_1 T_1}}{\pi_0^{(1)}} (\mu_1 T_1)^k.$$

We define $\bar{T}_1 \coloneqq 1/\mu_1$ and estimate for $T_1 \in \left(0, \bar{T}_1\right)$ with $q \coloneqq \mu_1 T_1 < 1$

$$\begin{aligned}\sum_{k=1}^{\infty} p_k^{(1)} \mathbb{E}[R_2 \mid B_k] \leq& \sum_{k=1}^{\infty} \frac{e^{\lambda_1 T_1}}{\pi_0^{(1)}} (\mu_1 T_1)^k C \left(k^2 + 1\right)\\ \leq& \frac{C e^{\frac{\lambda_1}{\mu_1}}}{\pi_{\inf}(\bar{T}_1)} \left(\sum_{k=1}^{\infty} k^2 q^k + \sum_{k=1}^{\infty} q^k\right)\\ =& \frac{C e^{\frac{\lambda_1}{\mu_1}}}{\pi_{\inf}(\bar{T}_1)} \left(\frac{q(1+q)}{(1-q)^3} + \frac{q}{1-q}\right),\end{aligned}$$

where we used Lemmas 9 and 10 in the first two lines and limits of geometric series in the last line. Finally, we are done because the term in the last line converges to zero for $T_1 \to 0$. □

Proof of Theorem 5: We treat the inequality $\Delta\bar{D} < 0$, which can be rearranged into

$$\frac{1}{r_0 + f_0}\left(\frac{r_0^{(2)}}{2} + \frac{\rho_2}{\rho_0} f_1\left(r_1 + \widetilde{r}_2 + w_1\right) + \frac{\rho_1}{\rho_0} f_2\left(\widetilde{r}_1 + r_2 + w_2\right)\right) < \frac{r_0^{(2)}}{2r_0}.$$

Strategy III. We can proceed in an analogous manner as in the proof of Theorem 3. Using the symmetry $\rho_1 = \rho_2$, we obtain the necessary and sufficient condition

$$\frac{r_0^{(2)}}{r_0^2} - \frac{1-\rho_1}{1-\rho_0} > 0$$

for 'it is worth waiting' at both stations with $T_1 = T_2 > 0$.

Strategies II and IV. In addition to the approach in the proofs above, we have to extend Lemma 10 by setting $T_1 = T_2 > 0$. Then, for a totally symmetric polling model, we get the necessary and sufficient condition

$$\frac{r_0^{(2)}}{r_0\left(r_1 + \widetilde{r}_2^{\mathrm{IV}}\right)} > 1 \tag{1.29}$$

for 'it is worth waiting' at both stations in the same way. We can conclude that (1.29) is satisfied if and only if the switchover times are non-deterministic, where we make use of $\widetilde{r}_2^{\mathrm{IV}} \le r_2$. To prove this last-mentioned inequality, we define $R \coloneqq \lambda_1 R_2$ and $r \coloneqq \mathbb{E}R$. We observe

$$r_2 = \mathbb{E}R_2 = \frac{\mathbb{E}R}{\lambda_1} = \frac{r}{\lambda_1}$$

and recall

$$\widetilde{r}_2^{\mathrm{IV}} = \mathbb{E}[R_2 \mid B_0] = \frac{\mathbb{E}\left[Re^{-R}\right]}{\lambda_1 \mathbb{E}\left[e^{-R}\right]}.$$

from (1.21) and (1.18). Therefore, it remains to show

$$\mathbb{E}\left[Re^{-R}\right] \le r\,\mathbb{E}\left[e^{-R}\right]. \tag{1.30}$$

We obtain

$$\begin{aligned}
\mathbb{E}\left[Re^{-R}\right] - r\,\mathbb{E}\left[e^{-R}\right] &= \mathbb{E}\left[Re^{-R} - re^{-R}\right] \\
&= \mathbb{E}\left[(R-r)e^{-R}\right] \\
&\le \mathbb{E}\left[(R-r)e^{-r}\right] \qquad\qquad (1.31)\\
&= e^{-r}\,\mathbb{E}\left[R-r\right] = 0,
\end{aligned}$$

where we used in (1.31) that the inequality

$$(x-r)e^{-x} \le (x-r)e^{-r}$$

holds for all $x \in \mathbb{R}$. This can be seen easily by a case distinction for $x \ge r$ and $x < r$. Finally, we can deduce (1.30) and this finishes the proof. $\square$

1.4 Conclusion

We briefly summarise the results of this chapter and list some open problems. We provided formulas for the mean average queueing delay in a polling model with Strategies II–IV, which can be computed for exponentially distributed service times. In the case of Strategy IV, we extended the work [10] to deterministic timers at *both* stations. Furthermore, we specified necessary and sufficient conditions for 'it is worth waiting' in comparison to the exhaustive strategy. The benefit of waiting arises from the asymmetry of the system or from non-deterministic switchover times.

By inspection of the derivation in Subsection 1.3.2, the restriction to exponentially distributed message lengths is only necessary for Strategies II and III but not for Strategy IV. This is due to the fact that we require time-dependent quantities to analyse the delay in a polling model with Strategies II and III. We note that the formula for the probability $P_{j,k}(x)$ that the queue length of an M/G/1 queue is k at time x given that the queue length is j at time zero only exists for the M/M/1 queue in the literature.

Regarding the comparison of the wait-and-see strategies among each other, we conjecture that Strategy III always yields a lower minimal delay compared to Strategy I and that Strategy II is the best of the four investigated wait-and-see strategies. This observation is based on a discussion of the variance of sojourn times. We recall that the comparison to Strategy IV depends on the particular choice of the system parameters of the polling model.

Finally, we make some suggestions for future work. In our case, the delay is weighted by the traffic intensity. However, the analysis of further definitions of queueing delays are possible, for instance, one may consider other weighting coefficients such that the impact of the stations changes. This idea is related to the notion of fairness among the stations, i.e., one might be interested in preventing monopolising the service at a certain station. This can also be implemented by introducing some priority mechanisms in the service policies. Of course, it is also natural to investigate other performance measures of the polling models. We think of queue lengths, some measures for fairness, and quantities that do not only cover expectations of waiting times but also take variances into account. We note that our approach, which uses a decomposition principle, is not applicable for that. Nevertheless, we emphasise that we were able to obtain exact results in this work, whereas in many cases only approximations are achievable.

In this chapter, we often deal with exponential distributions because the so-called memorylessness of such a distribution makes some techniques practicable. It would also be interesting to get results for other processes of message arrivals (more application-oriented arrival processes like in real-world communication systems) instead of restricting ourselves to Poisson processes. In addition, there are plenty of possibilities how modifications of the polling model can look like, for instance, queues might have only limited buffer sizes.

Furthermore, one can define extended wait-and-see strategies where the server has additional information on the current queue status at the other stations or where the server may see into the near future such that it knows about the incoming traffic. We expect that such strategies can be even better than Strategy II. However, it is not intuitively clear how the server should behave (switching or waiting idly) in certain situations in order to minimise the delay.

Remark. Most parts of Chapter 1 will appear in the journal *Probability in the Engineering and Informational Sciences*. The accepted article entitled *Improving the performance of polling models using forced idle times* has not been allocated to an issue yet but it has already been published online on *Cambridge Core* (see [5]).

Chapter 2

Scaling limits for a random boxes model

2.1 Introduction

2.1.1 Model

Let $B(x, u)$ denote the two-dimensional rectangular box in $\mathbb{R}^2$ with centre at x and edge lengths u_i for $i = 1, 2$. We consider a family of rectangles $(B(X^{(j)}, U^{(j)}))_j$ in $\mathbb{R}^2$ (also referred to as boxes) generated by a Poisson point process $(X^{(j)}, U^{(j)})_j$ in $\mathbb{R}^2 \times \mathbb{R}_+^2$. Let N be a Poisson random measure with intensity measure given by

$$n(\mathrm{d}x, \mathrm{d}u) = \lambda \mathrm{d}x F(\mathrm{d}u),$$

where the intensity λ is a positive constant. The probability measure F on $\mathbb{R}_+^2$ is given by

$$F(\mathrm{d}u) = c_F f_1(u_1) f_2(u_2) \mathrm{d}u_1 \mathrm{d}u_2, \tag{2.1}$$

where $c_F > 0$ is the normalising constant and $f_i(u_i) \sim 1/u_i^{\gamma_i+1}$ as $u_i \to \infty$ for $i = 1, 2$ with $\gamma_1 \in (1, 2)$ and $\gamma_1 < \gamma_2$. Hence, we assume w.l.o.g. that the tail of the distribution of the length is heavier than that of the width. Moreover, we assume for the sake of convenience that we have $c_F = 1$ (because one could simply think that c_F is included in λ in the case of $c_F \neq 1$) and we write $f(y) \sim g(y)$ if $f(y)/g(y) \to 1$. We note that

$$\int_{\mathbb{R}_+} u_i f_i(u_i) \mathrm{d}u_i < \infty$$

for $i = 1, 2$, i.e., the expected length and the expected width (and thus area) of a box are finite.

We discuss random fields defined on certain spaces of signed measures. Let us denote by $\mathcal{M}_2$ the linear space of signed measures μ on $\mathbb{R}^2$ with finite total variation $\|\mu\| \coloneqq |\mu|(\mathbb{R}^2) < \infty$, where $|\mu|$ is the total variation measure of μ (see, e.g., [36, p. 116]). We are interested in the cumulative volume induced by the boxes and measured by $\mu \in \mathcal{M}_2$. Therefore, we define the random field $J \coloneqq (J(\mu))_\mu$ on $\mathcal{M}_2$ by

$$J(\mu) \coloneqq \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \mu(B(x,u)) N(\mathrm{d}x, \mathrm{d}u).$$

Since our purpose is to deal with centred random fields, we introduce the notation for the corresponding centred Poisson random measure $\widetilde{N} \coloneqq N - n$ and centred integral $\widetilde{J}(\mu) \coloneqq J(\mu) - \mathbb{E}J(\mu)$.

The goal of this chapter is to obtain scaling limits for the random field $\widetilde{J}$. By scaling, we mean that the length and the width of the boxes are shrinking to zero, i.e., the scaled edge lengths are ρu_i with scaling parameter $\rho \to 0$, and that the expected number of boxes is increasing, i.e., the intensity λ of the Poisson point process is tending to infinity as a function of ρ. The precise behaviour of $\lambda = \lambda(\rho) \to \infty$ is specified in the different scaling regimes below. Following the notational convention from above, we denote by $\widetilde{J}_\rho$ the centred random field corresponding to the Poisson random measure N_ρ with the modified intensity $\lambda_\rho \coloneqq \lambda(\rho)$ and scaled edge lengths, i.e., F_ρ is the image measure of F by the change $u \mapsto \rho u$.

2.1.2 Related work

A basic reference on limit theorems of Poisson integrals is *Random processes by example* by Lifshits [29]. The main references for us are [7, 22].

Kaj et al. [22] study the limits of a spatial random field generated by independently and uniformly scattered random sets in $\mathbb{R}^d$. The sets (also referred to as grains) have a random volume but a predetermined shape. The size of a grain is given by a *single* heavy-tailed distribution, i.e., scaling means that the intensity λ grows to infinity while the mean volume ρ of the sets tends to zero. They obtain three different limits depending on the relative speed at which λ and ρ are scaled. Furthermore, they provide statistical properties of the limits.

In [7], Biermé et al. consider a random balls model of germ-grain type as well. The predetermined shape of the grains is a ball, whose size depends on the scaling parameter ρ and the random radius. The radius distribution has a power-law behaviour either in zero or at infinity, i.e., they deal with zooming in and zooming out. As main result, they can construct all self-similar, translation and rotation invariant Gaussian fields through zooming procedures in the random balls model.

Breton and Dombry [11] investigate weighted random balls models. The balls additionally have random weights, whose law belongs to the normal domain of attraction of the α-stable distribution with $\alpha \in (1, 2]$. They obtain different limiting random fields depending on the regimes and give statistical properties.

Gobard [20] considers a weighted random balls model with some dependence between the centres and the radii of the balls. On top of that, the underlying Poisson point process has an inhomogeneous intensity. Once again, the focus is on the asymptotic behaviour of the total mass while zooming.

An anisotropic scaling is examined by Pilipauskaitė and Surgailis [34]. They study the scaling limits of the random grain model on the plane with heavy-tailed grain area distribution. The anisotropy is implemented by scaling the x- and y-direction at different rates. Therefore, in the case of the grains being rectangles, the ratio of the edge lengths of *all* rectangles tends either to zero or to infinity under the scaling. This property distinguishes their model from our random boxes model, where each rectangle has a random length-to-width ratio that does not change under the scaling.

There is much more related work in a broader sense. We recall that [29] contains plenty of references, especially in connection with ‘teletraffic’ models. For instance, cluster Poisson processes (that can model the arrivals of data files) are discussed by Fasen [17] and Faÿ et al. [18]. The cluster Poisson process is a generalisation of the infinite source Poisson model. More precisely, at each Poisson point a cluster of packets (with heavy-tailed cluster sizes and within-cluster interarrival times) is initiated. In particular, the covariance structure is studied in [18] and the convergence of the (cumulative) input process is extensively analysed in [17].

2.1.3 Overview

In a nutshell, we extend the work in [7] and [22] to a random boxes model where the size of a grain depends on *two* differently heavy-tailed distributed random variables instead of just one random variable for the volume of the grain. To be more precise, the shape of the grains is rectangular with a random length and a random width (mutually independent). Therefore, our model differs from those in that the volume is given by the product of the length and the width, and each box simultaneously comes with a random length-to-width ratio. As a consequence, one main novelty is that our random boxes model leads to a greater number of scaling regimes than other random balls models (e.g., [7, 11, 22]). In particular, the so-called Poisson-lines scaling regime with its distinctive graphical representation has not arisen so far (see Section 2.5). The class of limiting random fields contains linear random fields that are Gaussian, compensated Poisson integrals and integrals with respect to a stable random measure.

Let us outline different scaling regimes that result in different limits. As mentioned above, the scaling regimes are defined by the joint behaviour of the scaling parameter ρ and the intensity λ_ρ of the Poisson point process as $\rho \to 0$.

We distinguish the following regimes:

- High intensity regime: $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to \infty$.
- Intermediate intensity regime: $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to a \in (0,\infty)$.
- Low intensity regime: $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to 0$.

The low intensity regime has to be divided once more into three different sub-regimes. Our naming of these sub-regimes is based on the limits and the objects spotted in a graphical representation (cf. Section 2.5). We distinguish the following sub-regimes:

- Gaussian-lines scaling regime: $\lambda_\rho \rho^{\gamma_1+\eta} \to a \in (0,\infty)$ for some constant $\eta \in (0,\gamma_2)$ and thus $\lambda_\rho \rho^{\gamma_1} \to \infty$. With regard to the scaling limit, it is of no importance to take care of the precise behaviour of $\lambda_\rho \rho^{\gamma_2}$ (as long as $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to 0$).
- Poisson-lines scaling regime: $\lambda_\rho \rho^{\gamma_1} \to a \in (0,\infty)$ and thus $\lambda_\rho \rho^{\gamma_2} \to 0$.
- Points scaling regime: $\lambda_\rho \rho^{\gamma_1} \to 0$.

So far, we have assumed $\gamma_1 < 2$. For $2 < \gamma_1 \leq \gamma_2$, the length and the width of the boxes have finite variances. In this case, there is only one scaling limit and we just require that $\lambda_\rho \to \infty$ as $\rho \to 0$, i.e., there is no further condition on the joint behaviour of ρ and λ_ρ.

2.2 Results

The following results are theorems of convergence of the finite-dimensional distributions of the centred and renormalised random field

$$\left(\frac{\widetilde{J}_\rho(\mu)}{n_\rho}\right)_{\mu\in\mathcal{M}}$$

to a limiting random field, where the corresponding space of signed measures $\mathcal{M}$ and the function $n_\rho := n(\rho)$ are defined in the theorems below, respectively. We denote this convergence by $\frac{\widetilde{J}_\rho(\cdot)}{n_\rho} \xrightarrow{\mathcal{M}} W(\cdot)$, where in each case the limiting random field $(W(\mu))_\mu$ is specified there.

2.2.1 High intensity regime

We look at the high intensity regime where $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to \infty$. First, we define the space of signed measures $\mathcal{M}^{\gamma_1,\gamma_2}$ where the theorem of convergence holds.

Definition 11. Let $\mathcal{M}^{\gamma_1,\gamma_2}$ be the subset of $\mathcal{M}_2$ with the following property: For each $\mu \in \mathcal{M}^{\gamma_1,\gamma_2}$, there exist constants $C > 0$ and α_i with $\gamma_i < \alpha_i \leq 2$ for $i = 1, 2$ such that the inequality

$$\int_{\mathbb{R}^2} \mu(B(x,u))^2 \mathrm{d}x \leq C \min(u_1, u_1^{\alpha_1}) \min(u_2, u_2^{\alpha_2}) \tag{2.2}$$

holds for all $u \in \mathbb{R}_+^2$.

The limiting random field in the high intensity regime is given by a centred Gaussian linear random field.

Theorem 12. *Let $\gamma_i \in (1,2)$ for $i = 1, 2$, $\lambda_\rho \to \infty$ and $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to \infty$ as $\rho \to 0$. Then, we have*

$$\frac{\widetilde{J}_\rho(\cdot)}{\sqrt{\lambda_\rho \rho^{\gamma_1+\gamma_2}}} \xrightarrow{\mathcal{M}^{\gamma_1,\gamma_2}} Z(\cdot)$$

as $\rho \to 0$, where $(Z(\mu))_\mu$ is the centred Gaussian linear random field with covariance function

$$C_Z(\mu, \nu) = \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \mu(B(x,u))\nu(B(x,u)) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}(x,u). \tag{2.3}$$

2.2.2 Intermediate intensity regime

In the intermediate intensity regime where $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to a \in (0,\infty)$, the space of signed measures is identical with the one in the high intensity regime. The limiting random field consists of compensated Poisson integrals.

Theorem 13. *Let $\gamma_i \in (1,2)$ for $i = 1, 2$, $\lambda_\rho \to \infty$ and $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to 1$ as $\rho \to 0$. Then, we have*

$$\widetilde{J}_\rho(\cdot) \xrightarrow{\mathcal{M}^{\gamma_1,\gamma_2}} J_I(\cdot)$$

as $\rho \to 0$, where $(J_I(\mu))_\mu$ is the linear random field of compensated Poisson integrals

$$J_I(\mu) := \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \mu(B(x,u)) \widetilde{N}_I(\mathrm{d}x, \mathrm{d}u),$$

where the intensity measure is given by

$$n_I(\mathrm{d}x, \mathrm{d}u) = \mathrm{d}x \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u_1 \mathrm{d}u_2.$$

We refer to Remark 35 below for the result in the (general) intermediate intensity regime with $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to a \in (0,\infty)$ as $\rho \to 0$, where a not necessarily equals 1.

2.2.3 Low intensity regime

The low intensity regime is defined by $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to 0$, which is divided once more into three different sub-regimes. In these sub-regimes, we additionally have to assume that the density function of the length of a box for *small* values is bounded, i.e., we assume that there is a constant $c_{f_1} > 0$ such that the inequality

$$f_1(u_1) \leq \frac{c_{f_1}}{u_1^{\gamma_1+1}} \tag{2.4}$$

holds for *all* $u_1 \in \mathbb{R}_+$. This technical assumption ensures the existence of a majorant for f_1 in the proofs below. From now on, we treat the three sub-regimes separately.

Gaussian-lines scaling regime

We define the space of signed measures $\mathcal{M}_L$ for the Gaussian-lines scaling regime where $\lambda_\rho \rho^{\gamma_1+\eta} \to a \in (0, \infty)$ for some constant $\eta \in (0, \gamma_2)$.

Definition 14. Let $\mathcal{M}_L$ be the subset of $\mathcal{M}_2$ where

- each $\mu \in \mathcal{M}_L$ has a density function f_μ, i.e., $\mu(\mathrm{d}x) = f_\mu(x)\mathrm{d}x$,
- for each $\mu \in \mathcal{M}_L$ the density function f_μ is bounded and decays at least exponentially fast, i.e., there exist constants $C_\mu > 0$ and $c_\mu > 0$ such that the inequality

$$|f_\mu(x)| \leq C_\mu e^{-c_\mu(|x_1|+|x_2|)} \tag{2.5}$$

  holds for all $x \in \mathbb{R}^2$,
- for each $\mu \in \mathcal{M}_L$ the pointwise convergence

$$\frac{1}{\varepsilon} \int_{B(x,(\frac{u_1}{\varepsilon}))} f_\mu(y)\mathrm{d}y \to \int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1, x_2)\mathrm{d}y_1 \tag{2.6}$$

  as $\varepsilon \to 0$ holds for all $(x, u_1) \in \mathbb{R}^2 \times \mathbb{R}_+$.

In the Gaussian-lines scaling regime, we require a further condition on the 'lighter' tail index, namely $\gamma_2 > 2$. Consequently, the width of a box has a finite variance. The limiting random field is a centred Gaussian linear random field.

Theorem 15. *Let $\gamma_1 \in (1, 2)$, $\gamma_2 > 2$, $\lambda_\rho \to \infty$ and $\lambda_\rho \rho^{\gamma_1+\eta} \to 1$ for some constant $\eta \in (0, \gamma_2)$ as $\rho \to 0$. Then, we have*

$$\frac{\widetilde{J}_\rho(\cdot)}{\rho^{1-\eta/2}} \xrightarrow{\mathcal{M}_L} Y(\cdot)$$

as $\rho \to 0$, where $(Y(\mu))_\mu$ is the centred Gaussian linear random field with covariance function $C_Y(\mu,\nu)$ given by

$$\int_{\mathbb{R}^2\times\mathbb{R}_+^2}\int_{\left[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}\right]^2} f_\mu(y_1,x_2)f_\nu(y_2,x_2)\mathrm{d}y\frac{u_2^2f_2(u_2)}{u_1^{\gamma_1+1}}\mathrm{d}(x,u). \tag{2.7}$$

We refer to Remark 37 below for the result in the (general) Gaussian-lines scaling regime with $\lambda_\rho\rho^{\gamma_1+\eta} \to a \in (0,\infty)$ as $\rho \to 0$, where a not necessarily equals 1.

Poisson-lines scaling regime

In the Poisson-lines scaling regime where $\lambda_\rho\rho^{\gamma_1} \to a \in (0,\infty)$, we provide the theorem of convergence to a random field consisting of compensated Poisson integrals. The corresponding space of signed measures coincides with the one from the Gaussian-lines scaling regime.

Theorem 16. *Let $\gamma_1 \in (1,2)$, $\gamma_1 < \gamma_2$, $\lambda_\rho \to \infty$ and $\lambda_\rho\rho^{\gamma_1} \to 1$ as $\rho \to 0$. Then, we have*

$$\frac{\widetilde{J}_\rho(\cdot)}{\rho} \xrightarrow{\mathcal{M}_L} J_L(\cdot)$$

as $\rho \to 0$, where $(J_L(\mu))_\mu$ is the linear random field of compensated Poisson integrals

$$J_L(\mu) := \int_{\mathbb{R}^2\times\mathbb{R}_+^2}\left(u_2\int_{\left[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}\right]} f_\mu(y_1,x_2)\mathrm{d}y_1\right)\widetilde{N}_L(\mathrm{d}x,\mathrm{d}u), \tag{2.8}$$

where the intensity measure is given by

$$n_L(\mathrm{d}x,\mathrm{d}u) = \mathrm{d}x\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1f_2(u_2)\mathrm{d}u_2. \tag{2.9}$$

We refer to Remark 36 below for the result in the (general) Poisson-lines scaling regime with $\lambda_\rho\rho^{\gamma_1} \to a \in (0,\infty)$ as $\rho \to 0$, where a not necessarily equals 1.

Remark 17. One can also view the compensated Poisson integral $J_L(\mu)$ in (2.8) from a slightly different perspective. It can be taken as if it originates from a *weighted* random balls model (more precisely, balls are lines) where the balls additionally have random weights and only the lengths tend to zero under the scaling. For this, let $(X^{(j)},U^{(j)})_j$ be an enumeration of the points of the Poisson random measure N_L with intensity measure given in (2.9), and interpret $X^{(j)}$, $U_1^{(j)}$ and $U_2^{(j)}$ as the centre of a line, the length of this line and the weight of this line, respectively. In our random boxes model these weights result from a 'transformation' of the widths of the boxes.

Points scaling regime

In the points scaling regime where $\lambda_\rho \rho^{\gamma_1} \to 0$, we investigate the scaling behaviour of $\widetilde{J}_\rho$ on the space of signed measures $\mathcal{M}_P$ which is given as follows:

Definition 18. Let $\mathcal{M}_P$ be the subset of $\mathcal{M}_2$ where

- each signed measure $\mu \in \mathcal{M}_P$ has a *continuous* density function f_μ, i.e., $\mu(\mathrm{d}x) = f_\mu(x)\mathrm{d}x$,
- for each $\mu \in \mathcal{M}_P$ the density function f_μ is bounded and decays at least exponentially fast, i.e., there exist constants $C_\mu > 0$ and $c_\mu > 0$ such that the inequality

$$|f_\mu(x)| \le C_\mu e^{-c_\mu(|x_1|+|x_2|)}$$

  holds for all $x \in \mathbb{R}^2$.

The limiting random field consists of integrals with respect to an α-stable random measure. For $\alpha \in (1,2)$, we denote by Λ_α the independently scattered α-stable random measure with unit skewness and Lebesgue control measure (see, e.g., [38]). We define the random linear functional

$$S_{\gamma_1}(\mu) \coloneqq \int_{\mathbb{R}^2} f_\mu(x)\Lambda_{\gamma_1}(\mathrm{d}x), \quad \text{for } \mu \in \mathcal{M}_P, \tag{2.10}$$

by its characteristic function at 1

$$\mathbb{E}\left(e^{iS_{\gamma_1}(\mu)}\right) = \exp\left(-\sigma_\mu^{\gamma_1}\left(1 - i\beta_\mu \tan\left(\frac{\pi\gamma_1}{2}\right)\right)\right),$$

where

$$\sigma_\mu = \|f_\mu\|_{\gamma_1}, \quad \beta_\mu = \|f_\mu\|_{\gamma_1}^{-\gamma_1}\left(\|f_{\mu+}\|_{\gamma_1}^{\gamma_1} - \|f_{\mu-}\|_{\gamma_1}^{\gamma_1}\right) \tag{2.11}$$

and $f_{\mu+} \coloneqq \max(f_\mu, 0)$, $f_{\mu-} \coloneqq -\min(f_\mu, 0)$.

Theorem 19. *Let $\gamma_1 \in (1,2)$, $\gamma_1 < \gamma_2$, $\lambda_\rho \to \infty$ and $\lambda_\rho\rho^{\gamma_1} \to 0$ as $\rho \to 0$. Then, we have*

$$\frac{\widetilde{J}_\rho(\cdot)}{c_{\gamma_1,\gamma_2}\lambda_\rho^{1/\gamma_1}\rho^2} \xrightarrow{\mathcal{M}_P} S_{\gamma_1}(\cdot)$$

as $\rho \to 0$, where the linear random field of functionals $(S_{\gamma_1}(\mu))_\mu$ and the constant c_{γ_1,γ_2} are defined in (2.10) *and* (2.48) *below, respectively.*

We emphasise that the 'heavier' tail index γ_1 for the length of a box appears primarily in the limit, i.e., the 'lighter' tail index γ_2 only enters into a constant. More precisely, the limit $S_{\gamma_1}(\mu)$ is a γ_1-stable random variable and the constant c_{γ_1,γ_2} given in (2.48) below is the only quantity depending on the tail index γ_2. This contrasts the limits in the high and intermediate intensity regimes, where both parameters γ_1 and γ_2 are present in a homogeneous way in each limit.

2.2.4 The finite variance case

We are interested in finding the scaling limit in the case where the area of a box has a finite variance. We assume that the length and the width of the boxes have finite second moments instead of heavy tails. Similar to above, let F be a probability measure on $\mathbb{R}_+^2$ given by

$$F(\mathrm{d}u) = f_1(u_1)f_2(u_2)\mathrm{d}u_1\mathrm{d}u_2.$$

Furthermore, we define

$$v_i := \int_{\mathbb{R}_+} u_i^2 f_i(u_i)\mathrm{d}u_i < \infty, \quad \text{for } i = 1, 2. \tag{2.12}$$

The following result shows that the centred and renormalised random field on the space $\mathcal{M}_P$ converges to a centred Gaussian linear random field. We stress that there does *not* exist a diversity of regimes to distinguish in the finite variance case. This case is also the much simpler case and the proof of this result can be viewed as a 'prototype proof' for all other regimes.

Theorem 20. *Let $\lambda_\rho \to \infty$ as $\rho \to 0$. Then, we have*

$$\frac{\widetilde{J}_\rho(\cdot)}{\rho^2\sqrt{\lambda_\rho v_1 v_2}} \xrightarrow{\mathcal{M}_P} X(\cdot)$$

as $\rho \to 0$, where v_i is defined in (2.12) *for $i = 1, 2$ and where $(X(\mu))_\mu$ is the centred Gaussian linear random field with covariance function*

$$C_X(\mu, \nu) = \int_{\mathbb{R}^2} f_\mu(x) f_\nu(x)\mathrm{d}x. \tag{2.13}$$

Remark 21. We note that two limiting random fields in all the preceding theorems of convergence have already arisen in identical form in related work. The centred Gaussian linear random field with covariance function given in (2.13) coincides with the corresponding one in the finite variance case of the random grain model where the size of a grain is given by a *single* distribution (see Theorem 1 in [22]). Moreover, the limiting random field consisting of integrals with respect to a stable random measure in the points scaling regime has also appeared there (cf. (13) in [22]). The index of stability is given by the index of the regularly varying tail of the volume of a grain there and by the 'heavier' tail index γ_1 for the length of a box in our random boxes model. All other limiting random fields seem to be new.

2.2.5 Statistical properties and extensions of the model

We start with providing some statistical properties of the different scaling limits Z, J_I, Y, J_L, S_{γ_1} and X.

Covariance. The covariance functions of the Gaussian random fields Z, Y and X are given in (2.3), (2.7) and (2.13), respectively. The covariance function of J_I in the intermediate intensity regime is exactly the same as in the high intensity regime (see (2.3)), but the limit J_I is not a Gaussian random field. In the points scaling regime, the scaling limit $S_{\gamma_1}(\mu)$ is γ_1-stable and thus does not have a finite variance. We distinguish two cases in the Poisson-lines scaling regime: If $\gamma_2 < 2$ holds, the compensated Poisson integral $J_L(\mu)$ does not have a finite variance. In contrast, if we assume $\gamma_2 > 2$, i.e., the width of a box has a finite variance, the scaling limit $J_L(\mu)$ has a finite variance as well and the covariance function coincides with the one in the Gaussian-lines scaling regime (see (2.7)).

Translation invariance. Let $s \in \mathbb{R}^2$. We define the translation of a signed measure $\tau_s\mu$ by $\tau_s\mu(A) := \mu(A - s)$ for any Borel set A. We call a random field W on $\mathcal{M}_W$ translation invariant if we have

$$(W(\tau_s\mu))_{\mu\in\mathcal{M}_W} = (W(\mu))_{\mu\in\mathcal{M}_W}$$

in finite-dimensional distributions for all $s \in \mathbb{R}^2$ ($\mathcal{M}_W$ has to be closed under translations τ_s). All limiting random fields Z, J_I, Y, J_L, S_{γ_1} and X are translation invariant on the respective spaces of signed measures.

Dilation. For all $a > 0$, the dilation of a signed measure μ_a is given by $\mu_a(A) := \mu(a^{-1}A)$ for any Borel set A. We call a random field W on $\mathcal{M}_W$ self-similar with index H if we have

$$(W(\mu_a))_{\mu\in\mathcal{M}_W} = (a^H W(\mu))_{\mu\in\mathcal{M}_W}$$

in finite-dimensional distributions for all $a > 0$ ($\mathcal{M}_W$ has to be closed under dilations μ_a).

The limiting Gaussian random fields Z, Y and X are self-similar with index $H = (2 - \gamma_1 - \gamma_2)/2$, $H = -\gamma_1/2$ and $H = -1$, respectively. In the points scaling regime, the limit S_{γ_1} is self-similar with index $H = 2/\gamma_1 - 2$. We emphasise that H is negative in these cases. If the reader expects H to be positive, a reason may be found in the way of defining the dilation of a signed measure which, however, is common in literature. One can also verify that the random field J_I in the intermediate intensity regime is not self-similar (cf. [22, p. 537]).

One calls a random field W with $\mathbb{E}W = 0$ on $\mathcal{M}_W$ (which has to be again closed under dilation) aggregate-similar (cf. [7, 21]) if there is a positive sequence $(a_m)_{m\geq 1}$ such that we have

$$(W(\mu_{a_m}))_{\mu\in\mathcal{M}_W} = \left(\sum_{k=1}^{m} W^k(\mu)\right)_{\mu\in\mathcal{M}_W}$$

in finite-dimensional distributions for all $m \geq 1$, where $(W^k)_{k\geq 1}$ are independent and identically distributed copies of W.

The random fields Z, Y, X, J_I and S_{γ_1} are aggregate-similar with $a_m = m^{1/(2-\gamma_1-\gamma_2)}$, $a_m = m^{-1/\gamma_1}$, $a_m = m^{-1/2}$, $a_m = m^{1/(2-\gamma_1-\gamma_2)}$ and $a_m = m^{1/(2-2\gamma_1)}$, respectively. Regarding the dilation in the Poisson-lines scaling regime, we mention that the scaling limit J_L only fulfils a modification of aggregate-similarity where the measure for the width is dilated simultaneously.

We continue with sketching feasible extensions of our random boxes model. For example, it is possible to allow non-negative σ-finite measures F instead of restricting ourselves to probability measures or to consider boxes (hyper-rectangles) in $\mathbb{R}^d$ with $d \geq 3$. Moreover, the model can be extended as follows:

Randomly rotated boxes. A modification of the random boxes model consists in additionally endowing the rectangles with independent and uniformly distributed orientations. We introduce the Haar measure $\mathrm{d}\theta$ on the group of rotations $SO(2)$ in $\mathbb{R}^2$ and consider the Poisson random measure N_ρ° on $\mathbb{R}^2 \times \mathbb{R}_+^2 \times SO(2)$ with intensity measure given by

$$n_\rho^\circ(\mathrm{d}x, \mathrm{d}u, \mathrm{d}\theta) = \lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u)\mathrm{d}\theta.$$

Then, the centred Poisson integral

$$\widetilde{J}_\rho^\circ(\mu) := \int_{\mathbb{R}^2\times\mathbb{R}_+^2\times SO(2)} \mu(B_\theta(x,u))\widetilde{N}_\rho^\circ(\mathrm{d}x, \mathrm{d}u, \mathrm{d}\theta)$$

is the object of interest, where $B_\theta(0,u) := \theta B(0,u)$ denotes the rectangle $B(0,u)$ rotated by θ and $B_\theta(x,u)$ for $x \neq 0$ is defined by

$$B_\theta(x,u) := x + B_\theta(0,u).$$

One can deduce analogous (rotation invariant) limiting random fields for this modified random boxes model as in Theorems 12, 13, 15, 16, 19 and 20. Since the probability measure $\mathrm{d}\theta$ on the group $SO(2)$ is not affected by the scaling as $\rho \to 0$, one can proceed as in the proofs there. One just has to change the spaces of signed measures slightly. Next, we define the rotation invariance for the sake of completeness.

Rotation invariance. Let $\theta \in SO(2)$. We define the rotation of a signed measure $\theta\mu$ by $\theta\mu(A) := \mu(\theta^{-1}A)$ for any Borel set A. We call a random field W on $\mathcal{M}_W$ rotation invariant if we have

$$(W(\theta\mu))_{\mu\in\mathcal{M}_W} = (W(\mu))_{\mu\in\mathcal{M}_W}$$

in finite-dimensional distributions for all $\theta \in SO(2)$. We note that $\mathcal{M}_W$ has to be closed under rotations $\theta\mu$.

2.3 Preliminaries and technical tools

From now on, we use c and C for constants that can differ from line to line.

2.3.1 Spaces of signed measures

We investigate the spaces of signed measures where the theorems of convergence in the high, intermediate and low intensity regimes hold, respectively. The following proposition ensures the linearity of these subspaces.

Proposition 22. *The subsets $\mathcal{M}^{\gamma_1,\gamma_2}$, $\mathcal{M}_L$ and $\mathcal{M}_P$ are linear subspaces of $\mathcal{M}_2$.*

Proof: We look at $\mathcal{M}^{\gamma_1,\gamma_2}$ first. If (2.2) holds for $\mu^{(k)}$ with $C^{(k)}$, $\alpha_i^{(k)}$ for $k = 1, 2$, then (2.2) holds for $\mu^{(1)}$ as well as for $\mu^{(2)}$ with the constants $C := \max\left(C^{(1)}, C^{(2)}\right)$ and $\alpha_i := \min\left(\alpha_i^{(1)}, \alpha_i^{(2)}\right)$ for $i = 1, 2$. This can be seen by a case distinction for $u_i \leq 1$ and $u_i > 1$. We thus obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left((a_1\mu^{(1)} + a_2\mu^{(2)})(B(x,u))\right)^2 \mathrm{d}x \\
&= \|(a_1\mu^{(1)} + a_2\mu^{(2)})(B(\cdot,u))\|_2^2 \\
&\leq \left(|a_1| \cdot \|\mu^{(1)}(B(\cdot,u))\|_2 + |a_2| \cdot \|\mu^{(2)}(B(\cdot,u))\|_2\right)^2 \qquad (2.14)\\
&\leq \Big(|a_1|(C\min(u_1, u_1^{\alpha_1})\min(u_2, u_2^{\alpha_2}))^{1/2} \\
&\qquad + |a_2|(C\min(u_1, u_1^{\alpha_1})\min(u_2, u_2^{\alpha_2}))^{1/2}\Big)^2 \\
&= (|a_1| + |a_2|)^2 C \min(u_1, u_1^{\alpha_1})\min(u_2, u_2^{\alpha_2})
\end{aligned}$$

for all $a_1, a_2 \in \mathbb{R}$, where we used the Minkowski inequality in (2.14). Therefore, we can conclude that $a_1\mu^{(1)} + a_2\mu^{(2)} \in \mathcal{M}^{\gamma_1,\gamma_2}$.

The fact that $\mathcal{M}_L$ and $\mathcal{M}_P$ are linear subspaces of $\mathcal{M}_2$ follows directly from Definitions 14 and 18 of these subspaces. $\square$

Remark 23. The linear space $\mathcal{M}^{\gamma_1,\gamma_2}$, where the theorems of convergence in the high and intermediate intensity regimes hold, is not yet the technically largest possible. We are able to weaken the condition in (2.2) as follows: For each $\mu \in \mathcal{M}^{\gamma_1,\gamma_2}$, there exist some constants $C > 0$, $\underline{\alpha}_i$ and $\overline{\alpha}_i$ with $0 < \underline{\alpha}_i < \gamma_i < \overline{\alpha}_i \leq 2$ for $i = 1, 2$ such that the inequality

$$\int_{\mathbb{R}^2} \mu(B(x,u))^2 \mathrm{d}x \leq C \min\left(u_1^{\underline{\alpha}_1}, u_1^{\overline{\alpha}_1}\right) \min\left(u_2^{\underline{\alpha}_2}, u_2^{\overline{\alpha}_2}\right)$$

holds for all $u \in \mathbb{R}_+^2$.

Remark 24. In Theorems 12 and 13 in the high and intermediate intensity regimes, we additionally assume $\gamma_2 < 2$ instead of just $\gamma_2 > \gamma_1$. The reason for that can be motivated in a natural way. On the one hand, we have to require that there exists some constant $\alpha_2 > \gamma_2$ in Definition 11 in order to prove the theorems of convergence. On the other hand, we want at least measures whose density functions have compact support to be contained in $\mathcal{M}^{\gamma_1,\gamma_2}$. As a consequence, $\alpha_2 \leq 2$ also has to be fulfilled. Therefore, both inequalities can only be satisfied simultaneously for $\gamma_2 < 2$.

Remark 25. We briefly comment on the characteristics of the spaces of signed measures in the low intensity sub-regimes (see Definitions 14 and 18). The assumption that each signed measure has a density function is obviously necessary since the density function appears explicitly in the limiting random fields. In contrast, we do not conjecture that the precise form of the technical assumption on the decay of the density function in (2.5) is necessary as well. Nevertheless, the reason for restricting the density functions to functions that decay at least exponentially fast is related to the maximal function of the signed measure given in (2.35) below. We have to ensure that Lemma 33 (ii) below holds in order to prove the theorems of convergence.

Next, we briefly touch on the comparison of these spaces of signed measures for $\gamma_i \in (1,2)$ for $i = 1,2$. We observe that the space $\mathcal{M}^{\gamma_1,\gamma_2}$ contains measures that do not have to have a density. Therefore, there exists some signed measure $\mu \in \mathcal{M}^{\gamma_1,\gamma_2}$, whereas $\mu \notin \mathcal{M}_k$ for $k \in \{L, P\}$. Conversely, we obtain the following result:

Proposition 26. *Let $\gamma_i \in (1,2)$ for $i = 1,2$. We have $\mathcal{M}_k \subseteq \mathcal{M}^{\gamma_1,\gamma_2}$ for $k \in \{L, P\}$.*

Proof: We recall that the density function of a signed measure in $\mathcal{M}_{L^{(k)}}$ for $k \in \{L, P\}$ satisfies for some constants $C_\mu > 0$ and $c_\mu > 0$

$$|f_\mu(x)| \leq C_\mu e^{-c_\mu(|x_1|+|x_2|)}$$

for all $x \in \mathbb{R}^2$. We have to show the validity of inequality (2.2), which can be reduced to the one-dimensional case. To be more precise, we can compute

$$\int_{\mathbb{R}} g_1(x_1,u_1)^2 \mathrm{d}x_1 \leq C \min\left(u_1, u_1^2\right) \tag{2.15}$$

for some $C > 0$ by a case distinction for $u_1 \leq 1$ and $u_1 > 1$, where

$$g_1(x_1,u_1) \coloneqq \int_{\left[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}\right]} e^{-c_\mu|y_1|} \mathrm{d}y_1. \tag{2.16}$$

First, we assume that $u_1 \leq 1$ and obtain

$$\begin{aligned}
&\int_{\mathbb{R}} g_1(x_1,u_1)^2 \mathrm{d}x_1 \\
&\leq 2\int_{\mathbb{R}_+}\left(\int_{[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}]} e^{-c_\mu y_1}\mathrm{d}y_1\right)^2 \mathrm{d}x_1 \\
&\leq 2\int_{\mathbb{R}_+}\left(\int_{[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}]} e^{-c_\mu (x_1-\frac{u_1}{2})}\mathrm{d}y_1\right)^2 \mathrm{d}x_1 \\
&= 2u_1^2 e^{c_\mu u_1}\int_{\mathbb{R}_+} e^{-2c_\mu x_1}\mathrm{d}x_1 \\
&= \frac{e^{c_\mu}}{c_\mu}u_1^2.
\end{aligned}$$

For $u_1 > 1$, we can estimate

$$\begin{aligned}
&\int_{\mathbb{R}} g_1(x_1,u_1)^2 \mathrm{d}x_1 \\
&\leq \int_{[-\frac{u_1}{2},\frac{u_1}{2}]}\left(\int_{\mathbb{R}} e^{-c_\mu|y_1|}\mathrm{d}y_1\right)^2 \mathrm{d}x_1 + 2\int_{(\frac{u_1}{2},\infty)} g_1(x_1,u_1)^2\mathrm{d}x_1 \\
&= \frac{4}{c_\mu^2}u_1 + \frac{2}{c_\mu^2}\int_{(\frac{u_1}{2},\infty)}\left(e^{-c_\mu x_1}\left(e^{c_\mu\frac{u_1}{2}} - e^{-c_\mu\frac{u_1}{2}}\right)\right)^2\mathrm{d}x_1 \\
&= \frac{4}{c_\mu^2}u_1 + \frac{2}{c_\mu^2}\left(e^{c_\mu\frac{u_1}{2}} - e^{-c_\mu\frac{u_1}{2}}\right)^2\int_{(\frac{u_1}{2},\infty)} e^{-2c_\mu x_1}\mathrm{d}x_1 \\
&\leq \frac{4}{c_\mu^2}u_1 + \frac{2}{c_\mu^2}\left(e^{c_\mu\frac{u_1}{2}}\right)^2\frac{1}{2c_\mu}e^{-c_\mu u_1} \\
&\leq \left(\frac{4}{c_\mu^2}+\frac{1}{c_\mu^3}\right)u_1.
\end{aligned}$$

Finally, we observe that (2.15) holds for all $u_1 \in \mathbb{R}_+$ with C given by

$$\frac{e^{c_\mu}}{c_\mu}+\frac{4}{c_\mu^2}+\frac{1}{c_\mu^3}$$

and we deduce the validity of inequality (2.2). □

2.3.2 Existence of the random fields

We deal with the existence of the random field $\widetilde{J}$ of interest and all the limiting random fields in the different scaling regimes. In this context, we cite a criterion for the existence of Poisson integrals from [24].

Lemma 27. *Let N be a Poisson random measure on some measurable space $(S,\mathcal{S})$ with intensity measure n. Then, for any measurable function f on S, we have* *(i)* $\int f \mathrm{d}N$ *exists if and only if* $\int \min(|f|, 1)\, \mathrm{d}n < \infty$*,*

(ii) $\int f \mathrm{d}(N-n)$ *exists if and only if* $\int \min\left(f^2, |f|\right) \mathrm{d}n < \infty$*.*

Proof: See Lemma 12.13 in [24]. □

We recall the notation $\widetilde{N} := N - n$ for the centred Poisson random measure. Applying Lemma 27, we can show that the random field J as well as the centred one $\widetilde{J}$ exist because we have

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2_+} |\mu(B(x,u))| n(\mathrm{d}x, \mathrm{d}u) \le \lambda \|\mu\| \int_{\mathbb{R}^2_+} u_1 u_2 F(\mathrm{d}u) < \infty.$$

To see this, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2} |\mu(B(x,u))| \lambda \mathrm{d}x F(\mathrm{d}u) \\
&= \lambda \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathbb{1}_{B(x,u)}(y) \mathrm{d}\mu(y) \right| \mathrm{d}x F(\mathrm{d}u) \\
&\le \lambda \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{1}_{\left\{|x_i - y_i| \le \frac{u_i}{2} \text{ for } i=1,2\right\}}(x,u,y) \mathrm{d}|\mu|(y) \mathrm{d}x F(\mathrm{d}u) \qquad (2.17) \\
&= \lambda \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2} \mathbb{1}_{\left\{|x_i| \le \frac{u_i}{2} \text{ for } i=1,2\right\}}(x,u) \mathrm{d}x F(\mathrm{d}u) \mathrm{d}|\mu|(y) \\
&= \lambda \int_{\mathbb{R}^2} \mathrm{d}|\mu|(y) \int_{\mathbb{R}^2_+} u_1 u_2 F(\mathrm{d}u) \\
&= \lambda |\mu|(\mathbb{R}^2) \int_{\mathbb{R}_+} u_1 f_1(u_1) \mathrm{d}u_1 \int_{\mathbb{R}_+} u_2 f_2(u_2) \mathrm{d}u_2 < \infty,
\end{aligned}$$

where we substituted $x_i = \widetilde{x}_i + y_i$ for $i = 1, 2$ in (2.17).

Furthermore, by standard facts on Poisson integrals (see, e.g., Section 7.4 in [29]) and Fubini's theorem, we note that the expected value of $J(\mu)$ is finite and given by

$$\mathbb{E}J(\mu) = \lambda \mu(\mathbb{R}^2) \int_{\mathbb{R}_+} u_1 f_1(u_1) \mathrm{d}u_1 \int_{\mathbb{R}_+} u_2 f_2(u_2) \mathrm{d}u_2.$$

Lemma 28. *We have*

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2_+} \mu(B(x,u))^2 \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}(x,u) < \infty$$

for all $\mu \in \mathcal{M}^{\gamma_1,\gamma_2}$*.*

Proof: This follows directly from Definition 11 of the space $\mathcal{M}^{\gamma_1,\gamma_2}$ by using the estimate in (2.2) for the function φ defined by

$$\varphi(u) := \int_{\mathbb{R}^2} \mu(B(x,u))^2 \mathrm{d}x \qquad (2.18)$$

for $u \in \mathbb{R}^2_+$. □

In the following, we briefly note that all the limiting random fields obtained in Theorems 12, 13, 15, 16, 19 and 20 are well-defined:

- Applying Lemma 28, one can check easily that the right hand side of (2.3) is a symmetric, positive-semidefinite function such that there is a centred Gaussian linear random field Z with covariance function given by (2.3).

- The existence of J_I follows from Lemmas 27 and 28.

- The proof of Theorem 15 shows that σ^2 given in (2.58) is finite. Hence, it can serve to construct the covariance function of a centred Gaussian linear random field Y.

- The existence of the compensated Poisson integral $J_L(\mu)$ for $\mu \in \mathcal{M}_L$ given in (2.8) can be verified by Lemma 27. One just has to show

$$\int_{\mathbb{R}^2\times\mathbb{R}^2_+} \min\left(|g(x,u)|, g(x,u)^2\right) \frac{1}{u_1^{\gamma_1+1}} f_2(u_2)\,\mathrm{d}(x,u) < \infty,$$

where

$$g(x,u) := u_2 \int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1,x_2)\mathrm{d}y_1.$$

This can be seen by a case distinction. Let us start with a general consideration. There is an $\varepsilon > 0$ such that

$$\min\left(|v|, v^2\right) \le \min\left(|v|^{\gamma_1-\varepsilon}, |v|^{\gamma_1+\varepsilon}\right) \tag{2.19}$$

with $1 < \gamma_1 - \varepsilon$ and $\gamma_1 + \varepsilon < \min(\gamma_2, 2)$. Furthermore, we observe

$$\begin{aligned} &\min\left(|ab|^{\gamma_1-\varepsilon}, |ab|^{\gamma_1+\varepsilon}\right) \\ \le&\min\left(|a|^{\gamma_1-\varepsilon}(|b|^{\gamma_1-\varepsilon}+|b|^{\gamma_1+\varepsilon}), |a|^{\gamma_1+\varepsilon}(|b|^{\gamma_1+\varepsilon}+|b|^{\gamma_1-\varepsilon})\right) \\ =&\min\left(|a|^{\gamma_1-\varepsilon}, |a|^{\gamma_1+\varepsilon}\right)(|b|^{\gamma_1-\varepsilon}+|b|^{\gamma_1+\varepsilon}). \end{aligned} \tag{2.20}$$

We use (2.19), (2.20) and the assumption (2.5) from Definition 14 to obtain

$$\begin{aligned} &\min\left(|g(x,u)|, g(x,u)^2\right) \\ \le& C\min\left(\left(g_1(x_1,u_1)e^{-c_\mu|x_2|}\right)^{\gamma_1-\varepsilon}, \left(g_1(x_1,u_1)e^{-c_\mu|x_2|}\right)^{\gamma_1+\varepsilon}\right) \\ &\times(|u_2|^{\gamma_1-\varepsilon}+|u_2|^{\gamma_1+\varepsilon}), \end{aligned} \tag{2.21}$$

where we recall the definition

$$g_1(x_1,u_1) := \int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} e^{-c_\mu|y_1|}\mathrm{d}y_1$$

from (2.16). Since

$$\int_{\mathbb{R}_+} (|u_2|^{\gamma_1-\varepsilon} + |u_2|^{\gamma_1+\varepsilon}) f_2\,(u_2)\,\mathrm{d}u_2 < \infty$$

due to $\gamma_1 + \varepsilon < \gamma_2$ and the asymptotic behaviour of f_2, and since

$$\int_{\mathbb{R}} e^{-c_\mu |x_2|(\gamma_1 \pm \varepsilon)} \mathrm{d}x_2 < \infty,$$

it remains to show that

$$\int_{\mathbb{R}\times\mathbb{R}_+} \min\left(g_1(x_1,u_1)^{\gamma_1-\varepsilon}, g_1(x_1,u_1)^{\gamma_1+\varepsilon}\right) \frac{1}{u_1^{\gamma_1+1}} \mathrm{d}(x_1,u_1) \qquad (2.22)$$

is finite. We can proceed analogously as in the proof of Proposition 26 in order to obtain for $u_1 \leq 1$

$$\int_{\mathbb{R}} g_1(x_1,u_1)^{\gamma_1+\varepsilon} \mathrm{d}x_1 \leq C u_1^{\gamma_1+\varepsilon}$$

and in the case of $u_1 \geq 1$

$$\int_{\mathbb{R}} g_1(x_1,u_1)^{\gamma_1-\varepsilon} \mathrm{d}x_1 \leq C u_1.$$

Finally, we can split the integral in (2.22) into two parts following this case distinction and see that these are bounded by

$$\int_{(0,1]} C u_1^{\gamma_1+\varepsilon} \frac{1}{u_1^{\gamma_1+1}} \mathrm{d}u_1 < \infty \quad \text{and} \quad \int_{(1,\infty)} C u_1 \frac{1}{u_1^{\gamma_1+1}} \mathrm{d}u_1 < \infty,$$

respectively. Therefore, the existence of the compensated Poisson integral $J_L(\mu)$ for $\mu \in \mathcal{M}_L$ is proven since the integral in (2.22) is finite. We note that in inequality (2.21) the particular exponent $\gamma_1 - \varepsilon$ is not required for this proof and one could also replace $\gamma_1 - \varepsilon$ by 1. However, we stick to the exponent $\gamma_1 - \varepsilon$ because we will need the estimates here for later purposes, for instance, in the proof of Theorem 16.

- Since $f_\mu \in L^{\gamma_1}(\mathbb{R}^2)$ for $\mu \in \mathcal{M}_P$, the random linear functional $S_{\gamma_1}(\mu)$ given in (2.10) is well-defined. We refer to Chapter 3 in [38] for this criterion and an extensive discussion on stable random processes.

- We can deduce from the proof of Theorem 20 that the integral in (2.13) is finite and serves to construct the covariance function of a centred Gaussian linear random field X.

2.3.3 Further useful lemmas

First, we define the function Ψ by

$$\Psi(v) := e^{iv} - 1 - iv, \quad \text{for } v \in \mathbb{R}, \tag{2.23}$$

which appears in the representation of the characteristic function of Poisson integrals below. Moreover, we often make use of the estimates in the following lemma:

Lemma 29. *We have for all $v \in \mathbb{R}$*

$$|\Psi(v)| \le \min\left(2|v|, \frac{v^2}{2}\right) \quad \text{and} \quad \left|\Psi(v) + \frac{v^2}{2}\right| \le \min\left(v^2, \frac{|v|^3}{6}\right).$$

Proof: See Lemma 1 in [22]. $\square$

We continue with some further lemmas that we use in the proofs of the results in Section 2.4.

Lemma 30. *Let F be a measure on $\mathbb{R}_+^2$ according to* (2.1) *and to the asymptotic behaviour specified there. Furthermore, let g be a continuous function on $\mathbb{R}_+^2$ such that there is a constant $C > 0$ for some $\alpha_i > \gamma_i$ for $i = 1, 2$ such that*

$$|g(u)| \le C \min(u_1, u_1^{\alpha_1}) \min(u_2, u_2^{\alpha_2}) \tag{2.24}$$

for all $u \in \mathbb{R}_+^2$. Then, we have

$$\int_{\mathbb{R}_+^2} g(u) F_\rho(\mathrm{d}u) \sim \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \tag{2.25}$$

as $\rho \to 0$.

Proof: The idea of the proof is as follows: We split the integral on the left hand side of (2.25) into four parts and treat the four integrals separately.

Let $\varepsilon > 0$ be given and define the constant c_0 by

$$c_0 \int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u = \left| \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \right|. \tag{2.26}$$

We note that $\int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u < \infty$ because of (2.24). (We refer to Remark 31 below for the special case of $\int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u = 0$.) Choose $N = N(\varepsilon)$ such that for all $u_i > N$ for $i = 1, 2$ we have

$$f_i(u_i) \le \frac{2}{u_i^{\gamma_i+1}} \tag{2.27}$$

and

$$\left| f_1(u_1) f_2(u_2) - \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \right| \le c_0 \frac{\varepsilon}{8} \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}}, \tag{2.28}$$

which is feasible due to the power-law assumption on the measure F. We write $\mathbb{R}_+^2 = \bigcup_{k=1}^4 \Omega_k$ with

$$\begin{aligned} \Omega_1 &\coloneqq (\rho N, \infty)^2, \\ \Omega_2 &\coloneqq (0, \rho N]^2, \\ \Omega_3 &\coloneqq (\rho N, \infty) \times (0, \rho N], \\ \Omega_4 &\coloneqq (0, \rho N] \times (\rho N, \infty). \end{aligned} \tag{2.29}$$

From now on, we discuss the four corresponding integrals separately.

1.) Using (2.28), we get

$$\begin{aligned} &\left| \int_{\Omega_1} g(u) F_\rho(\mathrm{d}u) - \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \right| \\ &\le \int_{\Omega_1} |g(u)| \left| f_1\left(\frac{u_1}{\rho}\right) \frac{1}{\rho} f_2\left(\frac{u_2}{\rho}\right) \frac{1}{\rho} - \rho^{\gamma_1+\gamma_2} \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \right| \mathrm{d}u \\ &\quad + \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2 \setminus \Omega_1} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \\ &\le c_0 \frac{\varepsilon}{8} \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \\ &\quad + \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2 \setminus \Omega_1} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \qquad (2.30) \\ &\le c_0 \frac{\varepsilon}{4} \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \end{aligned}$$

for ρ small enough, where we also used that the integral in (2.30) converges to zero by the dominated convergence theorem. Hence, we can deduce together with the definition of c_0 in (2.26) that there exists some $\rho_1 > 0$ such that for all $\rho < \rho_1$ we get

$$\begin{aligned} &\left| \frac{\int_{\Omega_1} g(u) F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u} - 1 \right| \\ &= \frac{\left| \int_{\Omega_1} g(u) F_\rho(\mathrm{d}u) - \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \right|}{\rho^{\gamma_1+\gamma_2} \left| \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \right|} \\ &\le \frac{c_0 \frac{\varepsilon}{4} \int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u}{\left| \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u \right|} \le \frac{\varepsilon}{4}. \end{aligned}$$

2.) We can show

$$\left|\int_{\Omega_2} g(u)F_\rho(\mathrm{d}u)\right| \in o(\rho^{\gamma_1+\gamma_2}).$$

Indeed, using (2.24), we obtain

$$\begin{aligned}\left|\int_{\Omega_2} g(u)F_\rho(\mathrm{d}u)\right| &\leq C\int_0^{\rho N}\int_0^{\rho N} u_1^{\alpha_1}u_2^{\alpha_2} f_1\left(\frac{u_1}{\rho}\right) f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho^2}\mathrm{d}u_1\mathrm{d}u_2\\ &= C\rho^{\alpha_1+\alpha_2}\int_0^N\int_0^N u_1^{\alpha_1}u_2^{\alpha_2} f_1(u_1)f_2(u_2)\mathrm{d}u_1\mathrm{d}u_2\\ &\leq C\rho^{\alpha_1+\alpha_2}N^{\alpha_1+\alpha_2}.\end{aligned}$$

Since $\alpha_1+\alpha_2 > \gamma_1+\gamma_2$, the assertion is true for $\rho \to 0$. More precisely, for ε and N as above there exists some $\rho_2 > 0$ such that for all $\rho < \rho_2$ we have

$$\left|\frac{\int_{\Omega_2} g(u)F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2}\int_{\mathbb{R}_+^2} g(u)\frac{1}{u_1^{\gamma_1+1}}\frac{1}{u_2^{\gamma_2+1}}\mathrm{d}u}\right| < \frac{\varepsilon}{4}.$$

3.) We prove

$$\left|\int_{\Omega_3} g(u)F_\rho(\mathrm{d}u)\right| \in o(\rho^{\gamma_1+\gamma_2}).$$

For N satisfying (2.27), we can observe

$$\begin{aligned}&\left|\int_{\Omega_3} g(u)F_\rho(\mathrm{d}u)\right|\\ &\leq \int_{\rho N}^{\infty}\int_0^{\rho N} |g(u)| f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho}\mathrm{d}u_2 f_1\left(\frac{u_1}{\rho}\right)\frac{1}{\rho}\mathrm{d}u_1\\ &\leq C\int_{\rho N}^{\infty}\int_0^{\rho N} \min\left(u_1, u_1^{\alpha_1}\right)\min\left(u_2, u_2^{\alpha_2}\right) f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho}\mathrm{d}u_2\frac{\rho^{\gamma_1}}{u_1^{\gamma_1+1}}\mathrm{d}u_1\\ &\leq C\rho^{\gamma_1}\int_{\rho N}^{\infty}\min\left(u_1, u_1^{\alpha_1}\right)\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1\int_0^{\rho N} u_2^{\alpha_2} f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho}\mathrm{d}u_2\\ &= C\rho^{\gamma_1}\rho^{\alpha_2}\int_0^N u_2^{\alpha_2} f_2(u_2)\mathrm{d}u_2 \leq C\rho^{\gamma_1+\alpha_2}N^{\alpha_2}.\end{aligned}$$

Since $\gamma_1+\alpha_2 > \gamma_1+\gamma_2$, we are done. In other words, for ε and N as above, there exists some $\rho_3 > 0$ such that for all $\rho < \rho_3$ we have

$$\left|\frac{\int_{\Omega_3} g(u)F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2}\int_{\mathbb{R}_+^2} g(u)\frac{1}{u_1^{\gamma_1+1}}\frac{1}{u_2^{\gamma_2+1}}\mathrm{d}u}\right| < \frac{\varepsilon}{4}.$$

4.) Proceeding analogously to 3.), one shows

$$\left|\int_{\Omega_4} g(u)F_\rho(\mathrm{d}u)\right| \in o(\rho^{\gamma_1+\gamma_2}).$$

Again, for ε and N as above, there exists some $\rho_4 > 0$ such that for all $\rho < \rho_4$ we obtain

$$\left| \frac{\int_{\Omega_4} g(u) F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u} \right| < \frac{\varepsilon}{4}.$$

Finally, we are able to deduce the assertion of the lemma: Let us define $\rho_0 := \min_{k \in \{1,\ldots,4\}} \rho_k$. Then, we obtain for all $\rho < \rho_0$, by splitting the domain of integration as mentioned above,

$$\begin{aligned}
&\left| \frac{\int_{\mathbb{R}_+^2} g(u) F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u} - 1 \right| \\
&\leq \left| \frac{\int_{\Omega_1} g(u) F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u} - 1 \right| + \sum_{k=2}^{4} \left| \frac{\int_{\Omega_k} g(u) F_\rho(\mathrm{d}u)}{\rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u} \right| \\
&\leq \frac{\varepsilon}{4} + \sum_{k=2}^{4} \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

where we used the results from the four parts above. □

Remark 31. We briefly sketch how to proceed in the proof of Lemma 30 in the special case of $\int_{\mathbb{R}_+^2} g(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u = 0$. We assume w.l.o.g. that

$$c_1 := \int_{\mathbb{R}_+^2} |g(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u > 0$$

(because otherwise the left hand side of (2.25) would already be zero). First, we mention that the assertion of the lemma reads as

$$\int_{\mathbb{R}_+^2} g(u) F_\rho(\mathrm{d}u) \in o(\rho^{\gamma_1+\gamma_2})$$

in the special case. Different from above, we can show in 1.) that

$$\left| \int_{\Omega_1} g(u) F_\rho(\mathrm{d}u) \right| \in o(\rho^{\gamma_1+\gamma_2}).$$

For this, choose $N = N(\varepsilon)$ such that for all $u_i > N$ for $i = 1, 2$ we have (2.27) and

$$\left| f_1(u_1) f_2(u_2) - \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \right| \leq \frac{\varepsilon}{2c_1} \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}}$$

instead of (2.28). Then, we can see by analogous estimates as in the beginning of 1.) above that

$$\left| \int_{\Omega_1} g(u) F_\rho(\mathrm{d}u) \right| \leq \varepsilon \rho^{\gamma_1+\gamma_2}$$

for ρ small enough. Together with the results in 2.), 3.) and 4.) in the proof above, we can deduce the assertion.

Lemma 32. *Let F be a measure on $\mathbb{R}_+^2$ according to* (2.1) *and to the asymptotic behaviour specified there. Furthermore, let (g_ρ) be a family of continuous functions on $\mathbb{R}_+^2$ with*

$$\lim_{\rho \to 0} \rho^{\gamma_1+\gamma_2} g_\rho(u) = 0$$

for all $u \in \mathbb{R}_+^2$ and

$$\rho^{\gamma_1+\gamma_2} |g_\rho(u)| \leq C \min(u_1, u_1^{\alpha_1}) \min(u_2, u_2^{\alpha_2})$$

for some constants $C > 0$ and $\alpha_i > \gamma_i$ for $i = 1, 2$ for all $u \in \mathbb{R}_+^2$. Then, we have

$$\lim_{\rho \to 0} \int_{\mathbb{R}_+^2} g_\rho(u) F_\rho(\mathrm{d}u) = 0. \tag{2.31}$$

Proof: The assumptions on g_ρ ensure that for all $\rho > 0$

$$\int_{\mathbb{R}_+^2} \rho^{\gamma_1+\gamma_2} |g_\rho(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u < \infty,$$

that there is an integrable majorant and that we get

$$\lim_{\rho \to 0} \int_{\mathbb{R}_+^2} \rho^{\gamma_1+\gamma_2} |g_\rho(u)| \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u = 0 \tag{2.32}$$

by the dominated convergence theorem.

Due to the power-law assumption on F, we can choose $N > 0$ such that for all $u_i > N$ for $i = 1, 2$ we have

$$f_i(u_i) \leq \frac{2}{u_i^{\gamma_i+1}}. \tag{2.33}$$

We use the same definition of the domains Ω_k for $k = 1, \ldots, 4$ as in (2.29) and continue discussing the corresponding four integrals separately. First, using (2.33) we get

$$\begin{aligned} \left| \int_{\Omega_1} g_\rho(u) F_\rho(\mathrm{d}u) \right| &\leq \int_{\rho N}^{\infty} \int_{\rho N}^{\infty} |g_\rho(u)| f_1\left(\frac{u_1}{\rho}\right) \frac{1}{\rho} f_2\left(\frac{u_2}{\rho}\right) \frac{1}{\rho} \mathrm{d}u_1 \mathrm{d}u_2 \\ &\leq \int_0^{\infty} \int_0^{\infty} \rho^{\gamma_1+\gamma_2} |g_\rho(u)| \frac{2}{u_1^{\gamma_1+1}} \frac{2}{u_2^{\gamma_2+1}} \mathrm{d}u_1 \mathrm{d}u_2. \end{aligned}$$

Therefore, we obtain together with (2.32) that

$$\lim_{\rho\to 0}\int_{\Omega_1} g_\rho(u)F_\rho(\mathrm{d}u) = 0.$$

Using the second assumption on g_ρ and (2.33), one can check that

$$\left|\int_{\Omega_k} g_\rho(u)F_\rho(\mathrm{d}u)\right| \to 0$$

as $\rho \to 0$ for $k = 2, 3, 4$ by proceeding analogously to the corresponding parts in the proof of Lemma 30. Combining all four partial results, we can deduce (2.31). $\square$

We introduce for a signed measure $\mu \in \mathcal{M}_k$ for $k \in \{L, P\}$ the local averages $m_\mu(x, u)$ by

$$m_\mu(x,u) \coloneqq \frac{1}{u_1 u_2}\int_{B(x,u)} f_\mu(y)\mathrm{d}y \tag{2.34}$$

and the maximal function m^*_μ by

$$m^*_\mu(x) \coloneqq \sup_{u\in\mathbb{R}^2_+}\frac{1}{u_1 u_2}\int_{B(x,u)} |f_\mu(y)|\mathrm{d}y. \tag{2.35}$$

Lemma 33. *Let $n_i(\rho) \to 0$ as $\rho \to 0$ for $i = 1, 2$.*

(i) For $\mu \in \mathcal{M}_P$, we have

$$\lim_{\rho\to 0} m_\mu\left(x, \begin{pmatrix} n_1(\rho)u_1 \\ n_2(\rho)u_2\end{pmatrix}\right) = f_\mu(x), \quad \text{for all } (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2_+.$$

(ii) Let $\beta > 1$. For $\mu \in \mathcal{M}_k$ for $k \in \{L, P\}$, there is a function $g \in L^\beta(\mathbb{R}^2)$ such that $m^*_\mu(x) \le g(x)$ for all $x \in \mathbb{R}^2$.

Proof:

(i) The assertion is true because the function f_μ is continuous and because there exists for all $\delta > 0$ some $\rho_0 > 0$ small enough such that the set $B\left(x, \begin{pmatrix} n_1(\rho)u_1 \\ n_2(\rho)u_2\end{pmatrix}\right)$ is contained in the ℓ^∞-ball with centre x and radius δ for all $\rho < \rho_0$.

More precisely, fix $x \in \mathbb{R}^2$, $u \in \mathbb{R}^2_+$ and let $\varepsilon > 0$. Since the function f_μ is continuous, there exists some $\delta = \delta(x, \varepsilon) > 0$ such that

$$|f_\mu(y) - f_\mu(x)| < \varepsilon \tag{2.36}$$

for all $y \in \mathbb{R}^2$ with $\|y - x\|_\infty < \delta$. Furthermore, for $i = 1, 2$, there exists some $\rho_i > 0$ such that $n_i(\rho) < \delta/u_i$ for all $\rho < \rho_i$. Let us define $\rho_0 := \min(\rho_1, \rho_2)$. Then, we obtain for all $\rho < \rho_0$

$$\begin{aligned}
&\left| m_\mu \left(x, \begin{pmatrix} n_1(\rho)u_1 \\ n_2(\rho)u_2 \end{pmatrix} \right) - f_\mu(x) \right| \\
&\leq \frac{1}{n_1(\rho)u_1 n_2(\rho)u_2} \int_{B\left(x, \begin{pmatrix} n_1(\rho)u_1 \\ n_2(\rho)u_2 \end{pmatrix}\right)} |f_\mu(y) - f_\mu(x)| \mathrm{d}y \\
&< \varepsilon,
\end{aligned}$$

as required, where we used (2.36) in the last line because

$$B\left(x, \begin{pmatrix} n_1(\rho)u_1 \\ n_2(\rho)u_2 \end{pmatrix} \right) \subseteq \left\{ y \in \mathbb{R}^2 \colon \|y - x\|_\infty < \delta \right\}.$$

(ii) We only make use of the assumption (2.5) on $\mu \in \mathcal{M}_k$ for $k \in \{L, P\}$ that the density function is bounded and decays at least exponentially fast. We obtain

$$\begin{aligned}
m_\mu^*(x) &\leq C_\mu \sup_{u \in \mathbb{R}_+^2} \frac{1}{u_1 u_2} \int_{B(x,u)} e^{-c_\mu |y_1|} e^{-c_\mu |y_2|} \mathrm{d}y \\
&= C_\mu \prod_{i=1,2} \sup_{u_i \in \mathbb{R}_+} \frac{1}{u_i} \int_{\left[x_i - \frac{u_i}{2}, x_i + \frac{u_i}{2}\right]} e^{-c_\mu |y_i|} \mathrm{d}y_i
\end{aligned} \tag{2.37}$$

and study the supremum in (2.37) by a case distinction. Let $x_i > 0$. We estimate

$$\begin{aligned}
&\sup_{u_i > 0} \frac{1}{u_i} \int_{\left[x_i - \frac{u_i}{2}, x_i + \frac{u_i}{2}\right]} e^{-c_\mu |y_i|} \mathrm{d}y_i \\
\leq &\sup_{0 < \frac{u_i}{2} \leq x_i} \frac{1}{u_i} \int_{\left[x_i - \frac{u_i}{2}, x_i + \frac{u_i}{2}\right]} e^{-c_\mu |y_i|} \mathrm{d}y_i \\
&+ \sup_{\frac{u_i}{2} \geq x_i} \frac{1}{u_i} \int_{\left[x_i - \frac{u_i}{2}, x_i + \frac{u_i}{2}\right]} e^{-c_\mu |y_i|} \mathrm{d}y_i
\end{aligned}$$

and treat the two terms on the right hand side of this inequality separately. For $0 < u_i/2 \leq x_i$, we get

$$\begin{aligned}
\frac{1}{u_i} \int_{\left[x_i - \frac{u_i}{2}, x_i + \frac{u_i}{2}\right]} e^{-c_\mu |y_i|} \mathrm{d}y_i &= \frac{1}{u_i} \frac{1}{c_\mu} \left(e^{-c_\mu \left(x_i - \frac{u_i}{2}\right)} - e^{-c_\mu \left(x_i + \frac{u_i}{2}\right)} \right) \\
&= \frac{e^{-c_\mu x_i}}{c_\mu} \frac{e^{\frac{c_\mu u_i}{2}} - e^{-\frac{c_\mu u_i}{2}}}{u_i} \\
&\leq \frac{e^{-c_\mu x_i}}{c} \frac{e^{c_\mu x_i} - e^{-c_\mu x_i}}{2x_i} \\
&\leq \frac{1}{2c_\mu x_i},
\end{aligned} \tag{2.38}$$

where we used in (2.38) the fact that the function

$$h(u_i) := \frac{e^{cu_i} - e^{-cu_i}}{u_i}$$

is increasing for $u_i \geq 0$. This can be seen by

$$\begin{aligned} h(u_i) &= \frac{1}{u_i}\left(\sum_{k=0}^{\infty}\frac{(cu_i)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-cu_i)^k}{k!}\right) \\ &= \frac{1}{u_i}\sum_{l=0}^{\infty}\frac{2(cu_i)^{2l+1}}{(2l+1)!} \\ &= 2\sum_{l=0}^{\infty}\frac{(cu_i)^{2l}}{(2l+1)!} \end{aligned}$$

because the last term is increasing in u_i. For $u_i/2 \geq x_i$, we observe

$$\frac{1}{u_i}\int_{\left[x_i-\frac{u_i}{2},x_i+\frac{u_i}{2}\right]} e^{-c_\mu|y_i|}\mathrm{d}y_i \leq \frac{1}{2x_i}\int_{\mathbb{R}} e^{-c_\mu|y_i|}\mathrm{d}y_i \leq \frac{1}{c_\mu x_i}.$$

Combining the estimates, we get

$$\sup_{u_i>0}\frac{1}{u_i}\int_{\left[x_i-\frac{u_i}{2},x_i+\frac{u_i}{2}\right]} e^{-c_\mu|y_i|}\mathrm{d}y_i \leq \frac{2}{c_\mu x_i}.$$

The corresponding estimate with $|x_i|$ for $x_i < 0$ follows directly because of symmetry. Furthermore, we can bound the supremum in (2.37) by

$$\sup_{u_i>0}\frac{1}{u_i}\int_{\left[x_i-\frac{u_i}{2},x_i+\frac{u_i}{2}\right]} e^{-c_\mu|y_i|}\mathrm{d}y_i \leq \sup_{u_i>0}\frac{1}{u_i}\int_{\left[x_i-\frac{u_i}{2},x_i+\frac{u_i}{2}\right]} 1\mathrm{d}y_i = 1.$$

Hence, we are able to conclude that $m_\mu^*(x) \leq g(x)$ for all $x \in \mathbb{R}^2$, where the function g is defined by

$$g(x) := C_\mu \prod_{i=1,2}\min\left(1, \frac{2}{c_\mu|x_i|}\right),$$

and we see that g^β is integrable with respect to x for any $\beta > 1$.

□

Remark 34. We briefly point out why the continuity condition of the density function f_μ is essential in the point scaling regime, in particular in Lemma 33 (i). If the boxes $B\left(x, \binom{n_1(\rho)u_1}{n_2(\rho)u_2}\right)$ had been *nicely shrinking sets* in the sense of [36, p. 140], the condition $f_\mu \in L^1(\mathbb{R}^2)$ would have been sufficient instead of requiring continuity (see Theorem 7.10 in [36]). In short, the crucial point for shrinking sets in order to be a sequence of nicely shrinking sets is that each set must occupy at least a certain portion of some spherical neighbourhood. For example, a shrinking grain in the random balls model where the size of a grain (with predetermined shape) depends only on a *single* distribution is nicely shrinking. In contrast, the boxes $B\left(x, \binom{n_1(\rho)u_1}{n_2(\rho)u_2}\right)$ in the proof of Theorem 19, where we apply Lemma 33 (i), are *not* nicely shrinking sets because the length-to-width ratio of the boxes tends to infinity there. Hence, we assume in Definition 18 that the density function f_μ is continuous such that Lemma 33 (i) holds.

2.4 Proofs

Due to the linearity of the mapping $\mu \mapsto \widetilde{J}_\rho(\mu)$ as well as the linearity of the limiting random fields Z, J_I, Y, J_L, S_{γ_1} and X, the convergence of the finite-dimensional distributions of the centred and renormalised versions of J_ρ is equivalent to the convergence of the one-dimensional distributions. This can be seen by using the Cramér-Wold device. Therefore, we only have to deal with the convergence of the characteristic function (w.l.o.g. at 1)

$$\mathbb{E}\exp\left(i\frac{\widetilde{J}_\rho(\mu)}{n_\rho}\right).$$

The strategy of the following proofs is similar to [7] and [22]. As already mentioned above, we use c and C for constants that can differ from line to line. Moreover, we often make use of the function Ψ defined in (2.23), in particular, in the representation of the characteristic function of $\widetilde{J}_\rho(\mu)$, which is given by

$$\mathbb{E}\left(e^{i\widetilde{J}_\rho(\mu)}\right) = \exp\left(\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi(\mu(B(x,u)))\lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u)\right).$$

We note that the order of the proofs differs from the order of the theorems presented in Section 2.2.

2.4.1 Intermediate intensity regime

Proof of Theorem 13: The characteristic function of $J_I(\mu)$ equals

$$\mathbb{E}\left(e^{iJ_I(\mu)}\right) = \exp\left(\int_{\mathbb{R}^2\times\mathbb{R}_+^2}\Psi(\mu(B(x,u)))\frac{1}{u_1^{\gamma_1+1}}\frac{1}{u_2^{\gamma_2+1}}\mathrm{d}(x,u)\right). \tag{2.39}$$

First, we define the function $\widetilde{\varphi}$ by

$$\widetilde{\varphi}(u) := \int_{\mathbb{R}^2}\Psi(\mu(B(x,u)))\mathrm{d}x, \quad \text{for } u \in \mathbb{R}_+^2.$$

One can verify similar to Lemma 6 in [22] that $\widetilde{\varphi}$ is continuous. Using $|\Psi(v)| \leq v^2$ from Lemma 29 and (2.2), there are constants $C > 0$ and α_i with $\gamma_i < \alpha_i \leq 2$ for $i = 1, 2$ such that

$$|\widetilde{\varphi}(u)| \leq C \min\left(u_1, u_1^{\alpha_1}\right) \min\left(u_2, u_2^{\alpha_2}\right).$$

Now, we apply Lemma 30 with $g := \widetilde{\varphi}$ to obtain

$$\int_{\mathbb{R}_+^2} \widetilde{\varphi}(u) F_\rho(\mathrm{d}u) \sim \rho^{\gamma_1+\gamma_2} \int_{\mathbb{R}_+^2} \widetilde{\varphi}(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u. \tag{2.40}$$

Using this and the scaling $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to 1$ shows the assertion. □

Remark 35. In the general case, let us say $\lambda_\rho \rho^{\gamma_1+\gamma_2} \to a^{2-\gamma_1-\gamma_2} \in (0, \infty)$ with $a > 0$ as $\rho \to 0$, the limiting compensated Poisson integral equals $J_I(\mu_a)$, where $\mu_a(\cdot) := \mu\left(a^{-1} \cdot\right)$. To see this, one can apply Theorem 13 to $\widetilde{J}'_\rho(\cdot)$, where $\lambda'_\rho := \lambda_\rho / a^{2-\gamma_1-\gamma_2}$. Then, one can deduce that the logarithm of the characteristic function of the limit in the general case equals

$$a^{2-\gamma_1-\gamma_2} \int_{\mathbb{R}_+^2} \widetilde{\varphi}(u) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u = a^2 \int_{\mathbb{R}_+^2} \widetilde{\varphi}\left(\frac{u}{a}\right) \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}u,$$

where the right hand side follows after the substitution $u_i = \widetilde{u}_i / a$ for $i = 1, 2$. Finally, we just observe

$$a^2 \widetilde{\varphi}\left(\frac{u}{a}\right) = a^2 \int_{\mathbb{R}^2} \Psi\left(\mu\left(B\left(x, \frac{u}{a}\right)\right)\right) \mathrm{d}x = \int_{\mathbb{R}^2} \Psi(\mu_a(B(x, u))) \mathrm{d}x.$$

2.4.2 High intensity regime

Proof of Theorem 12: For the sake of simplicity, we introduce

$$\varphi_\rho(u) := \int_{\mathbb{R}^2} \Psi\left(\frac{\mu(B(x, u))}{n_\rho}\right) \mathrm{d}x, \quad \text{for } u \in \mathbb{R}_+^2,$$

with $n_\rho := \sqrt{\lambda_\rho \rho^{\gamma_1+\gamma_2}}$ and recall that the characteristic function of $\frac{\widetilde{J}_\rho(\mu)}{n_\rho}$ is given by

$$\exp\left(\int_{\mathbb{R}_+^2} \varphi_\rho(u) \lambda_\rho F_\rho(\mathrm{d}u)\right).$$

The goal is to show the convergence of this characteristic function to

$$\exp\left(-\frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \mu(B(x, u))^2 \frac{1}{u_1^{\gamma_1+1}} \frac{1}{u_2^{\gamma_2+1}} \mathrm{d}(x, u)\right),$$

which corresponds to a centred Gaussian random variable. The covariance function given in (2.3) can then be obtained by the linearity of Z.

Since by assumption $n_\rho \to \infty$ as $\rho \to 0$, we know that $\Psi\left(\frac{\mu(B(x,u))}{n_\rho}\right)$ can be approximated by $-\frac{1}{2}\left(\frac{\mu(B(x,u))}{n_\rho}\right)^2$. To be more precise, we write

$$\int_{\mathbb{R}_+^2} \varphi_\rho(u)\lambda_\rho F_\rho(\mathrm{d}u) = -\frac{1}{2}\int_{\mathbb{R}_+^2} \varphi(u)\frac{\lambda_\rho}{n_\rho^2} F_\rho(\mathrm{d}u) + \int_{\mathbb{R}_+^2} \Delta_\rho(u) F_\rho(\mathrm{d}u), \quad (2.41)$$

where φ is given in (2.18) and

$$\begin{aligned}\Delta_\rho(u) &\coloneqq \varphi_\rho(u)\lambda_\rho + \frac{1}{2}\varphi(u)\frac{\lambda_\rho}{n_\rho^2} \\ &= \lambda_\rho \int_{\mathbb{R}^2}\left(\Psi\left(\frac{\mu(B(x,u))}{n_\rho}\right) + \frac{1}{2}\left(\frac{\mu(B(x,u))}{n_\rho}\right)^2\right)\mathrm{d}x.\end{aligned}$$

Applying Lemma 30 together with (2.2), the first integral on the right hand side of (2.41) converges to $\int_{\mathbb{R}_+^2} \varphi(u)\frac{1}{u_1^{\gamma_1+1}}\frac{1}{u_2^{\gamma_2+1}}\mathrm{d}u$. Here, we refer again to Lemma 6 in [22] in order to check the continuity of φ.

It remains to prove that the second integral on the right hand side of (2.41) converges to zero. For this purpose, we show that Δ_ρ satisfies the assumptions on g_ρ in Lemma 32.

First, we observe the estimates $\left|\Psi(v) + \frac{v^2}{2}\right| \le \frac{|v|^3}{6}$ and

$$\int_{\mathbb{R}^2} |\mu(B(x,u))|^3 \mathrm{d}x \le \|\mu\|^2 \int_{\mathbb{R}^2} |\mu(B(x,u))| \mathrm{d}x \le \|\mu\|^3 u_1 u_2.$$

In detail, the first estimate is given in Lemma 29. In order to check the second estimate, we just use

$$|\mu(B(x,u))| \le |\mu|(B(x,u)) \le |\mu|(\mathbb{R}^2) = \|\mu\|$$

for the first inequality and the second one holds since

$$\int_{\mathbb{R}^2} |\mu(B(x,u))| \mathrm{d}x \le \|\mu\| u_1 u_2,$$

which can be seen by following the lines in (2.17). Therefore, we obtain

$$\left|\rho^{\gamma_1+\gamma_2}\Delta_\rho(u)\right| = \left|\frac{n_\rho^2}{\lambda_\rho}\Delta_\rho(u)\right| \le \frac{\|\mu\|^3}{6}\frac{1}{n_\rho}u_1 u_2 \to 0$$

as $\rho \to 0$, which shows that the first assumption of Lemma 32 is satisfied. Using the second inequality in Lemma 29 and (2.2), the second assumption is also satisfied because we get

$$
\begin{aligned}
\rho^{\gamma_1+\gamma_2}\left|\Delta_\rho(u)\right| &= \left|\frac{n_\rho^2}{\lambda_\rho}\Delta_\rho(u)\right| \\
&\leq n_\rho^2 \int_{\mathbb{R}^2}\left(\frac{\mu(B(x,u))}{n_\rho}\right)^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2}\mu(B(x,u))^2 \mathrm{d}x \leq C\min\left(u_1, u_1^{\alpha_1}\right)\min\left(u_2, u_2^{\alpha_2}\right).
\end{aligned}
$$

□

2.4.3 Low intensity regime

Points scaling regime

Proof of Theorem 19: In a first step, we prove

$$
\lim_{\rho\to 0}\mathbb{E}\exp\left(i\frac{\widetilde{J}_\rho(\mu)}{\lambda_\rho^{1/\gamma_1}\rho^2}\right) = \exp\left(c_2^{\gamma_1}\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\Psi(u_1 f_\mu(x))\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1\mathrm{d}x\right),
$$

where c_2 is defined by

$$
c_2 := \left(\int_{\mathbb{R}_+} u_2^{\gamma_1} f_2(u_2)\mathrm{d}u_2\right)^{1/\gamma_1}. \tag{2.42}
$$

In a second step, we show that the right hand side is the characteristic function of an integral with respect to a stable random measure.

Step 1: We recall that the characteristic function of $\frac{\widetilde{J}_\rho(\mu)}{\lambda_\rho^{1/\gamma_1}\rho^2}$ equals

$$
\exp\left(\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi\left(\frac{1}{\lambda_\rho^{1/\gamma_1}\rho^2}\int_{B(x,u)} f_\mu(y)\mathrm{d}y\right)\lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u)\right). \tag{2.43}
$$

We use the definition of $m_\mu(x,u)$ in (2.34) and the density of the scaled measure F from (2.1) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi\left(\frac{1}{\lambda_\rho^{1/\gamma_1}\rho^2}\int_{B(x,u)} f_\mu(y)\mathrm{d}y\right)\lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \\
&= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi\left(\frac{u_1 u_2}{\lambda_\rho^{1/\gamma_1}\rho^2} m_\mu(x,u)\right)\lambda_\rho f_1\left(\frac{u_1}{\rho}\right)\frac{1}{\rho} f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho}\mathrm{d}x\mathrm{d}u \qquad (2.44) \\
&= \int_{\mathbb{R}^2\times\mathbb{R}_+^2}\Psi\left(u_1 m_\mu\left(x, \begin{pmatrix}\lambda_\rho^{1/\gamma_1}\rho\frac{u_1}{u_2} \\ \rho u_2\end{pmatrix}\right)\right)\frac{\lambda_\rho^{1+1/\gamma_1}}{u_2} f_1\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right) f_2(u_2)\mathrm{d}(x,u),
\end{aligned}
$$

where we substituted first $u_2 = \rho\widetilde{u}_2$ and then $u_1 = \lambda_\rho^{1/\gamma_1}\rho\frac{\widetilde{u}_1}{\widetilde{u}_2}$ in the last line. We note that

$$\lim_{\rho\to 0} m_\mu\left(x, \begin{pmatrix} \lambda_\rho^{1/\gamma_1}\rho\frac{u_1}{u_2} \\ \rho u_2 \end{pmatrix}\right) = f_\mu(x)$$

because of Lemma 33 (i) and that

$$\begin{aligned}
&\frac{\lambda_\rho^{1+1/\gamma_1}}{u_2} f_1\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right) \\
&= \frac{\lambda_\rho^{1+1/\gamma_1}}{u_2} f_1\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right)\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right)^{\gamma_1+1}\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right)^{-\gamma_1-1} \\
&= f_1\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right)\left(\lambda_\rho^{1/\gamma_1}\frac{u_1}{u_2}\right)^{\gamma_1+1}\frac{u_2^{\gamma_1}}{u_1^{\gamma_1+1}} \\
&\to \frac{u_2^{\gamma_1}}{u_1^{\gamma_1+1}}
\end{aligned}$$

as $\rho \to 0$ because of $\lambda_\rho^{1+1/\gamma_1} \to \infty$ and the asymptotic behaviour of f_1. Therefore, the integrand in the last line of (2.44) converges to

$$\Psi\left(u_1 f_\mu(x)\right)\frac{1}{u_1^{\gamma_1+1}} u_2^{\gamma_1} f_2(u_2).$$

If we can also find an integrable majorant of the integrand in the last line of (2.44), we obtain

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2}\Psi\left(\frac{1}{\lambda_\rho^{1/\gamma_1}\rho^2}\int_{B(x,u)} f_\mu(y)\mathrm{d}y\right)\lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \\
&= c_2^{\gamma_1}\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\Psi(u_1 f_\mu(x))\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1\mathrm{d}x
\end{aligned}$$

by the dominated convergence theorem, where c_2 is defined in (2.42). In order to find such a majorant, we use

$$|\Psi(v)| \leq 2\min\left(|v|, v^2\right) \tag{2.45}$$

from Lemma 29 and note that there is an $\varepsilon > 0$ with $1 < \gamma_1 - \varepsilon < \gamma_1 + \varepsilon < 2$ such that (2.19) and (2.20) hold. For all $\rho < \rho_0$ with ρ_0 small enough, the integrand (see last line of (2.44)) is therefore dominated by

$$2c_{f_1}\min\left(|u_1|^{\gamma_1-\varepsilon}, |u_1|^{\gamma_1+\varepsilon}\right)\left(|m_\mu^*(x)|^{\gamma_1-\varepsilon}+|m_\mu^*(x)|^{\gamma_1+\varepsilon}\right)\frac{u_2^{\gamma_1}}{u_1^{\gamma_1+1}} f_2(u_2), \tag{2.46}$$

where we also used the technical assumption in (2.4). Finally, we can see that (2.46) is integrable because of Lemma 33 (ii) and $1 < \gamma_1 - \varepsilon$.

Step 2: We look more closely at the integral

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\Psi(u_1 f_\mu(x))\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1\mathrm{d}x. \tag{2.47}$$

We split the integration over $\mathbb{R}^2$ into $\{x\colon f_\mu(x)\geq 0\}$ and $\{x\colon f_\mu(x)<0\}$ and note that $\Psi(0)=0$. We recall $f_{\mu+}:=\max(f_\mu,0)$ and $f_{\mu-}:=-\min(f_\mu,0)$. The substitution $\widetilde{u}_1=u_1 f_\mu(x)$ shows that (2.47) equals

$$d_{\gamma_1}\|f_{\mu+}\|_{\gamma_1}^{\gamma_1}+\bar{d}_{\gamma_1}\|f_{\mu-}\|_{\gamma_1}^{\gamma_1},$$

where $\bar{d}_{\gamma_1}$ is the complex conjugate of $d_{\gamma_1}:=\int_{\mathbb{R}_+}\Psi(u_1)\frac{1}{u_1^{\gamma_1+1}}\mathrm{d}u_1$. We obtain

$$d_{\gamma_1}=\frac{\Gamma(2-\gamma_1)}{\gamma_1(\gamma_1-1)}\cos\left(\frac{\pi\gamma_1}{2}\right)\left(1-i\tan\left(\frac{\pi\gamma_1}{2}\right)\right)$$

due to [38, p. 170]. Therefore, we can finally conclude that

$$\begin{aligned}
&\lim_{\rho\to 0}\log\mathbb{E}\exp\left(i\frac{\widetilde{J}_\rho(\mu)}{c_{\gamma_1,\gamma_2}\lambda_\rho^{1/\gamma_1}\rho^2}\right)\\
&=c_2^{\gamma_1}\left(d_{\gamma_1}\left\|\frac{f_{\mu+}}{c_{\gamma_1}c_2}\right\|_{\gamma_1}^{\gamma_1}+\bar{d}_{\gamma_1}\left\|\frac{f_{\mu-}}{c_{\gamma_1}c_2}\right\|_{\gamma_1}^{\gamma_1}\right)\\
&=-\left(\|f_{\mu+}\|_{\gamma_1}^{\gamma_1}+\|f_{\mu-}\|_{\gamma_1}^{\gamma_1}\right)+i\tan\left(\frac{\pi\gamma_1}{2}\right)\left(\|f_{\mu+}\|_{\gamma_1}^{\gamma_1}-\|f_{\mu-}\|_{\gamma_1}^{\gamma_1}\right)\\
&=-\sigma_\mu^{\gamma_1}\left(1-i\beta_\mu\tan\left(\frac{\pi\gamma_1}{2}\right)\right),
\end{aligned}$$

where

$$c_{\gamma_1,\gamma_2}:=c_{\gamma_1}c_2,\quad c_{\gamma_1}:=\left(-\frac{\Gamma(2-\gamma_1)}{\gamma_1(\gamma_1-1)}\cos\left(\frac{\pi\gamma_1}{2}\right)\right)^{1/\gamma_1}, \tag{2.48}$$

c_2 is given in (2.42) and σ_μ, β_μ are given in (2.11). □

Poisson-lines scaling regime

Proof of Theorem 16: We recall the characteristic function of $\frac{\widetilde{J}_\rho(\mu)}{\rho}$ given in (2.43). We proceed as in the preceding proof of Theorem 19. Using the definition of $m_\mu(x,u)$ in (2.34) and the density of the scaled measure F from (2.1), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi\left(\frac{1}{\rho}\int_{B(x,u)}f_\mu(y)\mathrm{d}y\right)\lambda_\rho\mathrm{d}xF_\rho(\mathrm{d}u)\\
&=\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2}\Psi\left(\frac{u_1u_2}{\rho}m_\mu(x,u)\right)\lambda_\rho f_1\left(\frac{u_1}{\rho}\right)\frac{1}{\rho}f_2\left(\frac{u_2}{\rho}\right)\frac{1}{\rho}\mathrm{d}x\mathrm{d}u\\
&=\int_{\mathbb{R}^2\times\mathbb{R}_+^2}\Psi\left(u_1u_2m_\mu\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)\frac{\lambda_\rho}{\rho}f_1\left(\frac{u_1}{\rho}\right)f_2(u_2)\,\mathrm{d}(x,u),
\end{aligned} \tag{2.49}$$

where we substituted $u_2 = \rho\widetilde{u}_2$ in (2.49). We note that due to (2.6) in Definition 14 of the space $\mathcal{M}_L$

$$\lim_{\rho\to 0} u_1 u_2 m_\mu\left(x, \left(\begin{smallmatrix} u_1 \\ \rho u_2 \end{smallmatrix}\right)\right) = u_2 \int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1, x_2)\mathrm{d}y_1$$

(pointwise for all $(x, u) \in \mathbb{R}^2 \times \mathbb{R}^2_+$) and that

$$\begin{aligned}\frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right) &= \frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right)\left(\frac{u_1}{\rho}\right)^{\gamma_1+1}\left(\frac{\rho}{u_1}\right)^{\gamma_1+1} \\ &= f_1\left(\frac{u_1}{\rho}\right)\left(\frac{u_1}{\rho}\right)^{\gamma_1+1}\lambda_\rho\rho^{\gamma_1}\frac{1}{u_1^{\gamma_1+1}} \\ &\to \frac{1}{u_1^{\gamma_1+1}}\end{aligned}$$

as $\rho \to 0$ because of $1/\rho \to \infty$, the asymptotic behaviour of f_1 and the fact that $\lambda_\rho\rho^{\gamma_1} \to 1$. Therefore, the integrand in (2.49) converges to

$$\Psi\left(u_2 \int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1, x_2)\mathrm{d}y_1\right)\frac{1}{u_1^{\gamma_1+1}} f_2(u_2). \tag{2.50}$$

If we can also find an integrable majorant of the integrand in (2.49), we obtain

$$\begin{aligned}&\lim_{\rho\to 0}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2}\Psi\left(\frac{1}{\rho}\int_{B(x,u)} f_\mu(y)\mathrm{d}y\right)\lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \\ &= \int_{\mathbb{R}^2\times\mathbb{R}^2_+}\Psi\left(u_2\int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1, x_2)\mathrm{d}y_1\right)\frac{1}{u_1^{\gamma_1+1}} f_2(u_2)\,\mathrm{d}(x, u) \quad (2.51)\end{aligned}$$

by the dominated convergence theorem. Using the estimates in (2.45) as well as in (2.19), an extended version of (2.20) and the technical assumption in (2.4), we see that the integrand in (2.49) is dominated by

$$\begin{aligned}&2\min\left(|u_1|^{\gamma_1-\varepsilon}, |u_1|^{\gamma_1+\varepsilon}\right)\left(|u_2|^{\gamma_1-\varepsilon} + |u_2|^{\gamma_1+\varepsilon}\right) \\ &\times\left(|m^*_\mu(x)|^{\gamma_1-\varepsilon} + |m^*_\mu(x)|^{\gamma_1+\varepsilon}\right)\frac{c_{f_1}}{u_1^{\gamma_1+1}} f_2(u_2)\end{aligned} \tag{2.52}$$

for all $\rho < \rho_0$ with ρ_0 small enough. Here, we have to choose $\varepsilon > 0$ such that $1 < \gamma_1 - \varepsilon$, $\gamma_1 + \varepsilon < 2$ as well as $\gamma_1 + \varepsilon < \gamma_2$. These conditions together with Lemma 33 (ii) ensure that (2.52) is integrable.

Since the characteristic function $\mathbb{E}\left(e^{iJ_L(\mu)}\right)$ of the limit $J_L(\mu)$ is given by the exponential of (2.51), the convergence of the characteristic function is proven. □

Remark 36. In the general case, let us say $\lambda_\rho \rho^{\gamma_1} \to a^{2-\gamma_1} \in (0,\infty)$ with $a > 0$ as $\rho \to 0$, we obtain $\frac{\widetilde{J}_\rho(\mu)}{a\rho} \to J_L(\mu_a)$, where we recall $\mu_a(\cdot) := \mu\left(a^{-1}\cdot\right)$. In order to prove this, we note that one gets (2.51) with the additional factor $a^{2-\gamma_1}$ for the logarithm of the characteristic function of the limit in the general case. After the substitution $x_1 = \widetilde{x}_1/a$, $x_2 = \widetilde{x}_2/a$, $u_1 = \widetilde{u}_1/a$ as well as $y_1 = \widetilde{y}_1/a$, we see that (2.51) equals

$$\int_{\mathbb{R}^2\times\mathbb{R}_+^2} \Psi\left(u_2 \int_{\left[x_1-\frac{u_1}{2},x_1+\frac{u_1}{2}\right]} \frac{1}{a} f_\mu(y_1/a, x_2/a)\mathrm{d}y_1\right) \frac{1}{u_1^{\gamma_1+1}} f_2\left(u_2\right) \mathrm{d}(x,u)$$

and we can deduce the result.

Gaussian-lines scaling regime

Proof of Theorem 15: We recall the characteristic function of $\frac{\widetilde{J}_\rho(\mu)}{\rho^{1-\eta/2}}$, which equals

$$\exp\left(\int_{\mathbb{R}^2\times\mathbb{R}_+^2} \Psi\left(\frac{\mu\left(B\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)}{\rho^{1-\eta/2}}\right) \frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right) f_2(u_2)\mathrm{d}(x,u)\right)$$

after the substitution $u_2 = \rho\widetilde{u}_2$. The goal is to show for some $\sigma^2 > 0$ the convergence of this characteristic function to $\exp\left(-\sigma^2/2\right)$, which corresponds to a centred Gaussian random variable.

To be more precise, we write

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{R}_+^2} \Psi\left(\frac{\mu\left(B\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)}{\rho^{1-\eta/2}}\right) \frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right) f_2(u_2)\mathrm{d}(x,u) \\ =&-\frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^2} u_2^2 \left(\frac{\mu\left(B\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)}{\rho u_2}\right)^2 \rho^\eta \frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right) f_2(u_2)\mathrm{d}(x,u) \quad (2.53)\\ &+\int_{\mathbb{R}^2\times\mathbb{R}_+^2} \Delta_\rho(u,x)\frac{\lambda_\rho}{\rho} f_1\left(\frac{u_1}{\rho}\right) f_2(u_2)\mathrm{d}(x,u), \quad (2.54)\end{aligned}$$

where

$$\Delta_\rho(u,x) := \Psi\left(\frac{\mu\left(B\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)}{\rho^{1-\eta/2}}\right) + \frac{1}{2}\left(\frac{\mu\left(B\left(x,\left(\begin{smallmatrix}u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)}{\rho^{1-\eta/2}}\right)^2. \quad (2.55)$$

First, we discuss the integral in (2.54) in the case of $\gamma_2 > 3$. Since we have $\left|\Psi(v) + \frac{v^2}{2}\right| \le \frac{|v|^3}{6}$ (see Lemma 29), we can bound (2.55) and thus can bound the integrand by

$$
\begin{aligned}
&\frac{1}{6}\rho^{3\eta/2}u_2^3\left(\frac{|\mu\left(B\left(x,\left(\frac{u_1}{\rho u_2}\right)\right)\right)|}{\rho u_2}\right)^3\frac{\lambda_\rho}{\rho}f_1\left(\frac{u_1}{\rho}\right)f_2(u_2)\\
&\le\frac{C_\mu^3}{6}\rho^{\eta/2}u_2^3\left(\frac{1}{\rho u_2}\int_{B\left(x,\left(\frac{u_1}{\rho u_2}\right)\right)}e^{-c_\mu(|y_1|+|y_2|)}\mathrm{d}y\right)^3\rho^\eta\frac{\lambda_\rho}{\rho}c_{f_1}\left(\frac{\rho}{u_1}\right)^{\gamma_1+1}f_2(u_2)\\
&\le C\rho^{\eta/2}u_2^3g_1(x_1,u_1)^3g_2(x_2)^3\frac{1}{u_1^{\gamma_1+1}}f_2(u_2),
\end{aligned}
\tag{2.56}
$$

for $\rho<\rho_0$ with ρ_0 small enough, where g_1 is given in (2.16) and

$$
g_2(x_2):=\min\left(1,\frac{2}{c_\mu|x_2|}\right).
$$

Here, we used the particular decay of the bounded density function f_μ in (2.5), the technical assumption in (2.4) and the scaling $\lambda_\rho\rho^{\gamma_1+\eta}\to 1$. Furthermore, we used

$$
\sup_{\rho>0}\frac{1}{\rho u_2}\int_{\left[x_2-\frac{\rho u_2}{2},x_2+\frac{\rho u_2}{2}\right]}e^{-c_\mu|y_2|}\mathrm{d}y_2\le g_2(x_2)
$$

from the proof of Lemma 33 (ii). By (2.56), we see that the integrand in (2.54) has an integrable majorant since we assumed $\gamma_2>3$ and because g_2^3 is integrable with respect to x_2 and $g_1(x_1,u_1)^3/u_1^{\gamma_1+1}$ is also integrable (in order to check this, one can follow the lines below (2.16) and (2.22)). Moreover, the majorant converges to zero because of $\rho^{\eta/2}\to 0$.

In the case of $2<\gamma_2\le 3$, we note that there is an $\varepsilon>0$ such that $2<\gamma_2-\varepsilon<3$ as well as $\left|\Psi(v)+\frac{v^2}{2}\right|\le|v|^{\gamma_2-\varepsilon}$. The last-mentioned estimate can be deduced from the second inequality in Lemma 29 by a case distinction (cf. (2.19)). Similar to above, we can bound the integrand in (2.54) by

$$
\begin{aligned}
&\rho^{(\gamma_2-\varepsilon)\eta/2}\left(\frac{|\mu\left(B\left(x,\left(\frac{u_1}{\rho u_2}\right)\right)\right)|}{\rho}\right)^{\gamma_2-\varepsilon}\frac{\lambda_\rho}{\rho}f_1\left(\frac{u_1}{\rho}\right)f_2(u_2)\\
&\le\rho^{(\gamma_2-\varepsilon-2+2)\eta/2}u_2^{\gamma_2-\varepsilon}\left(\frac{|\mu\left(B\left(x,\left(\frac{u_1}{\rho u_2}\right)\right)\right)|}{\rho u_2}\right)^{\gamma_2-\varepsilon}\frac{\lambda_\rho}{\rho}c_{f_1}\left(\frac{\rho}{u_1}\right)^{\gamma_1+1}f_2(u_2)\\
&\le C\rho^{(\gamma_2-\varepsilon-2)\eta/2}u_2^{\gamma_2-\varepsilon}\left(g_1(x_1,u_1)g_2(x_2)\right)^{\gamma_2-\varepsilon}\lambda_\rho\rho^{\gamma_1+\eta}\frac{1}{u_1^{\gamma_1+1}}f_2(u_2)\\
&\le C\rho^{(\gamma_2-\varepsilon-2)\eta/2}u_2^{\gamma_2-\varepsilon}g_1(x_1,u_1)^{\gamma_2-\varepsilon}g_2(x_2)^{\gamma_2-\varepsilon}\frac{1}{u_1^{\gamma_1+1}}f_2(u_2)
\end{aligned}
\tag{2.57}
$$

for $\rho<\rho_0$ with ρ_0 small enough. Using $\gamma_1<\gamma_2-\varepsilon$, we can see by (2.57) that the integrand in (2.54) has an integrable majorant because $g_2^{\gamma_2-\varepsilon}$ and $g_1(x_1,u_1)^{\gamma_2-\varepsilon}/u_1^{\gamma_1+1}$ are integrable (with the same reasons as above) and that it converges to zero because of $\gamma_2-\varepsilon-2>0$.

Therefore, we obtain in both cases that the integral in (2.54) converges to zero by the dominated convergence theorem.

Next, we deal with the integral in (2.53) and show that it converges to

$$\sigma^2 := \int_{\mathbb{R}^2\times\mathbb{R}_+^2} u_2^2 \left(\int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1,x_2)\mathrm{d}y_1 \right)^2 \frac{f_2(u_2)}{u_1^{\gamma_1+1}} \mathrm{d}(x,u). \tag{2.58}$$

The convergence of the integrand can be seen similar to above using Definition 14 of the space $\mathcal{M}_L$, the asymptotic behaviour of f_1 and the fact that $\lambda_\rho \rho^{\gamma_1+\eta} \to 1$. A majorant of the integrand is given by

$$C u_2^2 g_1(x_1,u_1)^2 g_2(x_2)^2 \frac{1}{u_1^{\gamma_1+1}} f_2(u_2),$$

which is integrable for $\gamma_2 > 2$. Applying the dominated convergence theorem, the convergence of the characteristic function is proven. By linearity, the covariance function given in (2.7) follows from (2.58). □

Remark 37. In the general case, let us say $\lambda_\rho \rho^{\gamma_1+\eta} \to a^2 \in (0,\infty)$ with $a > 0$ as $\rho \to 0$, the limit is a centred Gaussian linear random field which is given by $(Y(a\mu))_\mu$, where $a\mu$ has the density af_μ and the variance of $Y(a\mu)$ is just $a^2\sigma^2$. This can be seen easily since we obtain the additional factor a^2 in (2.58).

2.4.4 The finite variance case

Proof of Theorem 20: We use the definition of $m_\mu(x,u)$ in (2.34) to obtain for the logarithm of the characteristic function of $\frac{\widetilde{J}_\rho(\mu)}{\rho^2\sqrt{\lambda_\rho v_1 v_2}}$

$$\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2} \Psi\left(\frac{1}{\rho^2\sqrt{\lambda_\rho v_1 v_2}} \int_{B(x,u)} f_\mu(y)\mathrm{d}y \right) \lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \\
&= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2} \Psi\left(\frac{u_1u_2}{\rho^2\sqrt{\lambda_\rho v_1 v_2}} m_\mu(x,u) \right) \lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \\
&= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^2} \Psi\left(\frac{u_1u_2}{\sqrt{\lambda_\rho v_1 v_2}} m_\mu\left(x, \left(\begin{smallmatrix}\rho u_1\\ \rho u_2\end{smallmatrix}\right)\right) \right) \lambda_\rho \mathrm{d}x F(\mathrm{d}u),
\end{aligned}$$

where we substituted $u_2 = \rho\widetilde{u}_2$ and $u_1 = \rho\widetilde{u}_1$ in the last line. We note that

$$\lim_{\rho\to 0} m_\mu\left(x, \left(\begin{smallmatrix}\rho u_1\\ \rho u_2\end{smallmatrix}\right)\right) = f_\mu(x)$$

because of Lemma 33 (i). Due to the second inequality in Lemma 29 and $\lambda_\rho \to 0$, we get

$$\lim_{\rho\to 0} \Psi\left(\frac{u_1u_2}{\sqrt{\lambda_\rho v_1 v_2}} m_\mu\left(x, \left(\begin{smallmatrix}\rho u_1\\ \rho u_2\end{smallmatrix}\right)\right) \right) \lambda_\rho = -\frac{u_1^2 u_2^2 f_\mu(x)^2}{2 v_1 v_2}.$$

Furthermore, we use $|\Psi(v)| \leq \frac{v^2}{2}$ from Lemma 29 and the definition of $m_\mu^*(x)$ in (2.35) to obtain

$$\Psi\left(\frac{u_1 u_2}{\sqrt{\lambda_\rho v_1 v_2}} m_\mu\left(x, \left(\begin{smallmatrix}\rho u_1\\ \rho u_2\end{smallmatrix}\right)\right)\right)\lambda_\rho \leq \frac{u_1^2 u_2^2 m_\mu^*(x)^2}{2v_1 v_2}.$$

Since the right hand side can serve as an integrable majorant, we can apply the dominated convergence theorem and obtain

$$\lim_{\rho\to 0} \mathbb{E}\exp\left(i\frac{\widetilde{J}_\rho(\mu)}{\rho^2\sqrt{\lambda_\rho v_1 v_2}}\right) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^2} f_\mu(x)^2 \mathrm{d}x\right),$$

which is the characteristic function of a centred Gaussian random variable. Hence, we can conclude that the limiting random field is a centred Gaussian linear random field with covariance function given in (2.13). □

2.4.5 Statistical properties

We verify the facts on statistical properties in Subsection 2.2.5. Regarding translation and dilation, we briefly note that if the density function exists, we have $f_{\tau_s\mu}(\cdot) = f_\mu(\cdot - s)$ and $f_{\mu_a}(\cdot) = a^{-2} f_\mu(a^{-1}\cdot)$.

Covariance. The covariance function of J_I follows directly from the following fact on compensated Poisson integrals (cf. Section 7.4 in [29]): When f and g are square-integrable functions with respect to the intensity measure n, we have

$$\mathbb{E}\left(\int f\mathrm{d}\widetilde{N}\int g\mathrm{d}\widetilde{N}\right) = \int fg\mathrm{d}n. \tag{2.59}$$

We can also use this identity in the Poisson-lines scaling regime. We observe for $\gamma_2 > 2$ that

$$\int_{\mathbb{R}^2\times\mathbb{R}_+^2} u_2^2\left(\int_{\left[x_1-\frac{u_1}{2}, x_1+\frac{u_1}{2}\right]} f_\mu(y_1, x_2)\mathrm{d}y_1\right)^2 \frac{1}{u_1^{\gamma_1+1}} f_2(u_2)\mathrm{d}(x,u) \tag{2.60}$$

is finite (cf. finiteness of (2.58)). Therefore, $J_L(\mu)$ has a finite variance and the covariance function can be obtained using (2.59). However, in the case of $\gamma_2 < 2$, we note that (2.60) is not finite because $\int_{\mathbb{R}_+} u_2^2 f_2(u_2)\mathrm{d}u_2$ is not finite. This is due to the asymptotic behaviour of f_2 and we deduce that $J_L(\mu)$ does not have a finite variance.

In the points scaling regime, the integral $S_{\gamma_1}(\mu)$ with respect to an α-stable random measure with $\alpha = \gamma_1 < 2$ is an α-stable random variable and thus has an infinite second moment.

Translation invariance. For the Gaussian random fields, it suffices to show that the covariance functions of $(Z(\cdot))$, $(Y(\cdot))$ and $(X(\cdot))$ given in (2.3), (2.7) and (2.13) are identical to the ones of the Gaussian random fields $(Z(\tau_s\cdot))$, $(Y(\tau_s\cdot))$ and $(X(\tau_s\cdot))$, respectively. This follows directly from the translation invariance of the Lebesgue measure.

The argumentation for the random fields J_I, S_{γ_1} and J_L reads as follows: Due to the linearity of each random field, we can restrict ourselves to the one-dimensional distribution. We can observe that the translation does not change the characteristic function because of the translation invariance of the Lebesgue measure.

In other words, the random field $\widetilde{J}_\rho$ for all $\rho > 0$ is translation invariant (by the translation invariance of the Lebesgue measure) and this property thus also holds for the limits.

Dilation First, we discuss the self-similarity. Just like in the argumentation for the translation, we investigate the covariance function of the dilated Gaussian random field. Substituting $x = a\widetilde{x}$ and $u = a\widetilde{u}$ produces the factor $a^{2-\gamma_1-\gamma_2}$ compared to the covariance function of Z in the non-dilated case. In the Gaussian-lines scaling regime, we obtain the factor $a^{-\gamma_1}$ compared to the covariance function of Y after the substitution $x_2 = a\widetilde{x}_2$, $y = a\widetilde{y}$, $x_1 = a\widetilde{x}_1$ and $u_1 = a\widetilde{u}_1$. In the finite variance case, we just have to substitute $x = a\widetilde{x}$ and the factor then equals a^{-2}. Using the linearity of Z, Y and X leads to $H = (2 - \gamma_1 - \gamma_2)/2$, $H = -\gamma_1/2$ and $H = -1$, respectively.

In the points scaling regime, we substitute $x = a\widetilde{x}$ and $u_1 = a^2\widetilde{u}_1$ in the characteristic function (using (2.47) for the representation). Hence, the factor $a^{2-2\gamma_1}$ arises in front of the integral in (2.47). From a different perspective, the factor $(a^H)^{\gamma_1}$ appears at this point if we represent the characteristic function of $a^H S_{\gamma_1}(\mu)$ using the integral in (2.47). Therefore, we obtain $H = 2/\gamma_1 - 2$.

Next, we deal with the aggregate-similarity. We look at the Gaussian random fields first. We prove that centred Gaussian random fields which are self-similar with index H are aggregate-similar with $a_m = m^{1/(2H)}$. To show equality in finite-dimensional distributions, it suffices to show equality of the covariance function for a centred Gaussian random field. We recall that the sum of m independent and identically distributed centred Gaussian vectors is again a centred Gaussian vector where the covariance function is multiplied by m. Using the self-similarity, we obtain $a_m^{2H} = m$. Therefore, we conclude that $a_m = m^{1/(2H)} = m^{1/(2-\gamma_1-\gamma_2)}$, $a_m = m^{-1/\gamma_1}$ and $a_m = m^{-1/2}$ for the Gaussian random fields Z, Y and X, respectively.

In the intermediate intensity regime, we use the linearity of J_I, the fact that the characteristic function of m independent and identically distributed random vectors equals the single characteristic function to the power of m, and that the substitution $x = a_m\widetilde{x}$ and $u = a_m\widetilde{u}$ in the characteristic function of the dilated random field J_I produces the factor $a^{2-\gamma_1-\gamma_2}$ in front of the integral in (2.39). It follows that $a_m = m^{1/(2-\gamma_1-\gamma_2)}$ just like in the high intensity regime.

In the points scaling regime, we proceed as in the intermediate regime. Then, a substitution produces the factor $a^{2-2\gamma_1}$ and we get $a_m = m^{1/(2-2\gamma_1)}$.

2.5 Graphical representation

It is natural to ask if one can distinguish the sub-regimes in the low intensity regime due to their characteristics by a graphical representation. Hence, we ran simulations of Poisson point processes in the different scaling regimes for some small scaling parameter ρ and appropriate intensity λ_ρ. We generated random Poisson points such that the centres of the boxes are located in a bounded domain and we chose Pareto distributions for the length and the width of the boxes. Then, we plotted the boxes that are filled with black colour.

Before we present the graphical results, we introduce two quantities, which can characterise the different scaling regimes. The first quantity is the expected number of boxes with length and width greater than one that cover the origin. This quantity is given by

$$\int_{\mathbb{R}_+^2} \int_{\mathbb{R}^2} \mathbb{1}_{[1,\infty)^2}(u) \mathbb{1}_{(-\frac{u_1}{2},\frac{u_1}{2})}(x_1) \mathbb{1}_{(-\frac{u_2}{2},\frac{u_2}{2})}(x_2) \lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \sim c\lambda_\rho \rho^{\gamma_1+\gamma_2}$$

as $\rho \to 0$, where we applied Karamata's Theorem (see, e.g., [8, 35]) and where c is some positive constant. For instance, this expected number tends to infinity in the high intensity regime and to zero in the low intensity regime. Since the low intensity regime is divided once more into three different sub-regimes, we introduce a further crucial quantity: The expected number of boxes that cross the line $0 \times \mathbb{R}$, that have centres in the strip $\mathbb{R} \times [0,1]$ and that have lengths greater than one, is given by

$$\int_{\mathbb{R}_+^2} \int_{\mathbb{R}\times[0,1]} \mathbb{1}_{[1,\infty)}(u_1) \mathbb{1}_{(-\frac{u_1}{2},\frac{u_1}{2})}(x_1) \lambda_\rho \mathrm{d}x F_\rho(\mathrm{d}u) \sim c\lambda_\rho \rho^{\gamma_1}$$

as $\rho \to 0$, where c is again some positive constant.

Figure 2.1 consists of two samples for the Gaussian-lines scaling regime, which are generated by simulations. The left one represents the original model, whereas the right comes from the modified model where each box is additionally randomly rotated (cf. Subsection 2.2.5 for the definition of this modified model). We plotted the boxes of these samples, where different levels of grey arose due to the superposition of the (filled) boxes. Here, the more frequently a point is covered by a rectangle, the more darkly the point is plotted. We spot a kind of noise with horizontal lines in the sample on the left hand side. Referring to the crucial quantity defined in the preceding paragraph, we note that the scaling $\lambda_\rho \rho^{\gamma_1} \to \infty$ indicates *many* (long) horizontal lines. Depending on the behaviour of $\lambda_\rho \rho^{\gamma_2}$, one may also spot vertical lines. However, we do not spot vertical lines in this sample because we chose $\lambda_\rho \rho^{\gamma_2} \to 0$ as $\rho \to 0$ here.

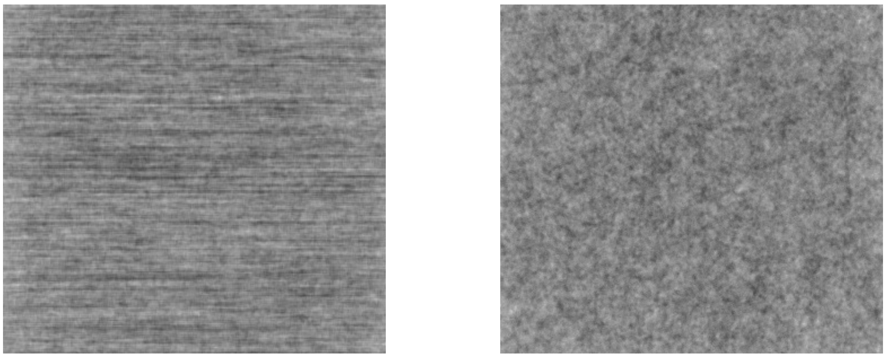

Figure 2.1: Gaussian-lines scaling regime.

In the Poisson-lines scaling regime, we have $\lambda_\rho \rho^{\gamma_1} \to a \in (0, \infty)$ and $\lambda_\rho \rho^{\gamma_2} \to 0$ as $\rho \to 0$. On the one hand, this indicates a different behaviour for the length and the width of the boxes. On the other hand, the intensity increases more slowly than in the Gaussian-lines scaling regime. This behaviour corresponds to the fact that the above-mentioned expected number of boxes with length greater than one that cross the line $0 \times \mathbb{R}$ converges to a positive constant and with width greater than one that cross the line $\mathbb{R} \times 0$ tends to zero.

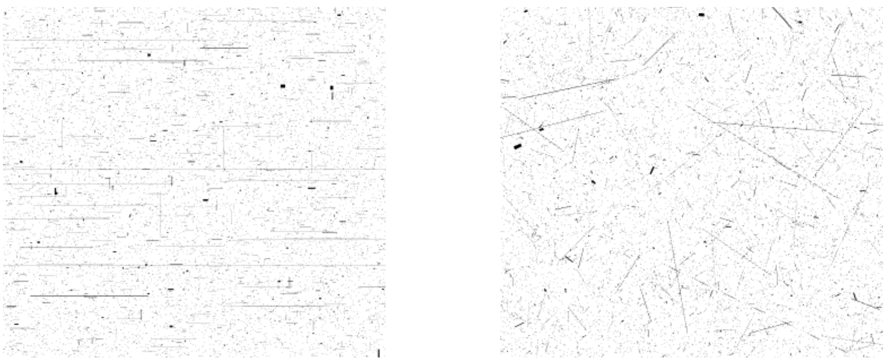

Figure 2.2: Poisson-lines scaling regime.

Two samples of the random boxes model in the Poisson-lines scaling regime are given in Figure 2.2. Besides points, we spot some horizontal lines in the sample on the left hand side. In the sample on the right hand side, each box is just additionally randomly rotated.

Finally, Figure 2.3 represents a sample of the random boxes model in the points scaling regime. The characteristic of the points scaling regime arises due to $\lambda_\rho \rho^{\gamma_i} \to 0$ as $\rho \to 0$ for $i = 1, 2$. One can only spot points instead of lines or boxes in this sample.

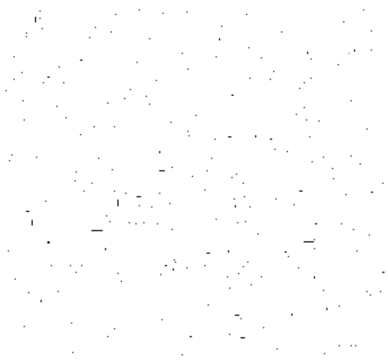

Figure 2.3: Points scaling regime.

2.6 Conclusion

We start with a brief summary of this chapter. We introduced a particular random boxes model in $\mathbb{R}^2$ where the length and the width of the rectangles are heavy-tailed distributed. Depending on the joint behaviour of the shrinking mean edge lengths and the increasing intensity of the underlying Poisson point process, we defined several scaling regimes. The regimes are identified by high, intermediate and low intensity. The latter one is divided once more into three sub-regimes, which we could also distinguish by a graphical representation. Following this classification, we were able to obtain different scaling limits, namely, linear random fields that are Gaussian, compensated Poisson integrals and integrals with respect to a stable random measure. At the end, we presented some statistical properties of the limits.

In connection with related random balls models, we extended the work in [7] and [22] to a random boxes model where the size of a grain depends on *two* differently heavy-tailed distributed random variables. As a consequence, we got a greater number of scaling regimes than in the above-mentioned references, whereas the class of limiting random fields contains the same 'species'. In detail, even in the low intensity regime in our work, which is divided into three sub-regimes, all three 'species' of the class of limits appear. In contrast, the corresponding scaling regime in [22], which is denoted by *small-grain scaling* there, shows only integrals with respect to a stable random measure as limit. Moreover, we noticed that the tail indices γ_1 and γ_2 enter in different ways into the limit in each low intensity sub-regime, whereas both indices are present in a homogeneous way in each limit in the high and intermediate intensity regimes.

Finally, we mention some unsolved problems and make suggestions for future work. From the technical point of view, one might ask how some assumptions can be weakened. Especially, we think of conditions that characterise the respective spaces of signed measures. Nevertheless, this does not lead to different scaling limits in the considered cases. It would also be interesting to investigate the scaling behaviour in borderline cases, which are omitted in this work. For instance, in the low intensity regime with $\gamma_1 = \gamma_2$, it is not clear whether a limit exists and if the 'species' of the limit, in the case of existence, has already appeared in the class of limiting random fields so far.

Furthermore, extensions of the random boxes model in a broader sense are possible. As seen in related work in Section 2.1.2, one can consider additional weights for the boxes, an inhomogeneous intensity of the Poisson random measure, or some dependence between random variables. For example, one may introduce dependences between the location and the size of a box or between the length and the width of a box.

Remark. Most parts of Chapter 2 are also available on *arXiv* in the preprint entitled *Scaling limits for a random boxes model* (see [6]).

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